IGCSE · Cambridge (CIE) · Further Maths

1 hour · 18 questions

Exam Questions

Equations, Inequalities & Graphs

Modulus Functions / Graphs of Cubic Polynomials

Medium (7 questions)	/30
Hard (6 questions)	/24
Very Hard (5 questions)	/22
Total Marks	**/76**

Medium Questions

1 Solve the equation $5|5x - 2| - 1 = 14$.

Answer

First isolate the modulus

$$5|5x-2| = 15$$
$$|5x-2| = 3$$

To obtain the first solution, assume the contents of the modulus brackets are positive and solve

$$5x - 2 = 3$$
$$5x = 5$$

$x = 1$ [1]

To obtain the second solution, assume the contents of the modulus brackets are negative

$$5x - 2 = -3 \quad (\text{or } -5x + 2 = 3)$$

[1]

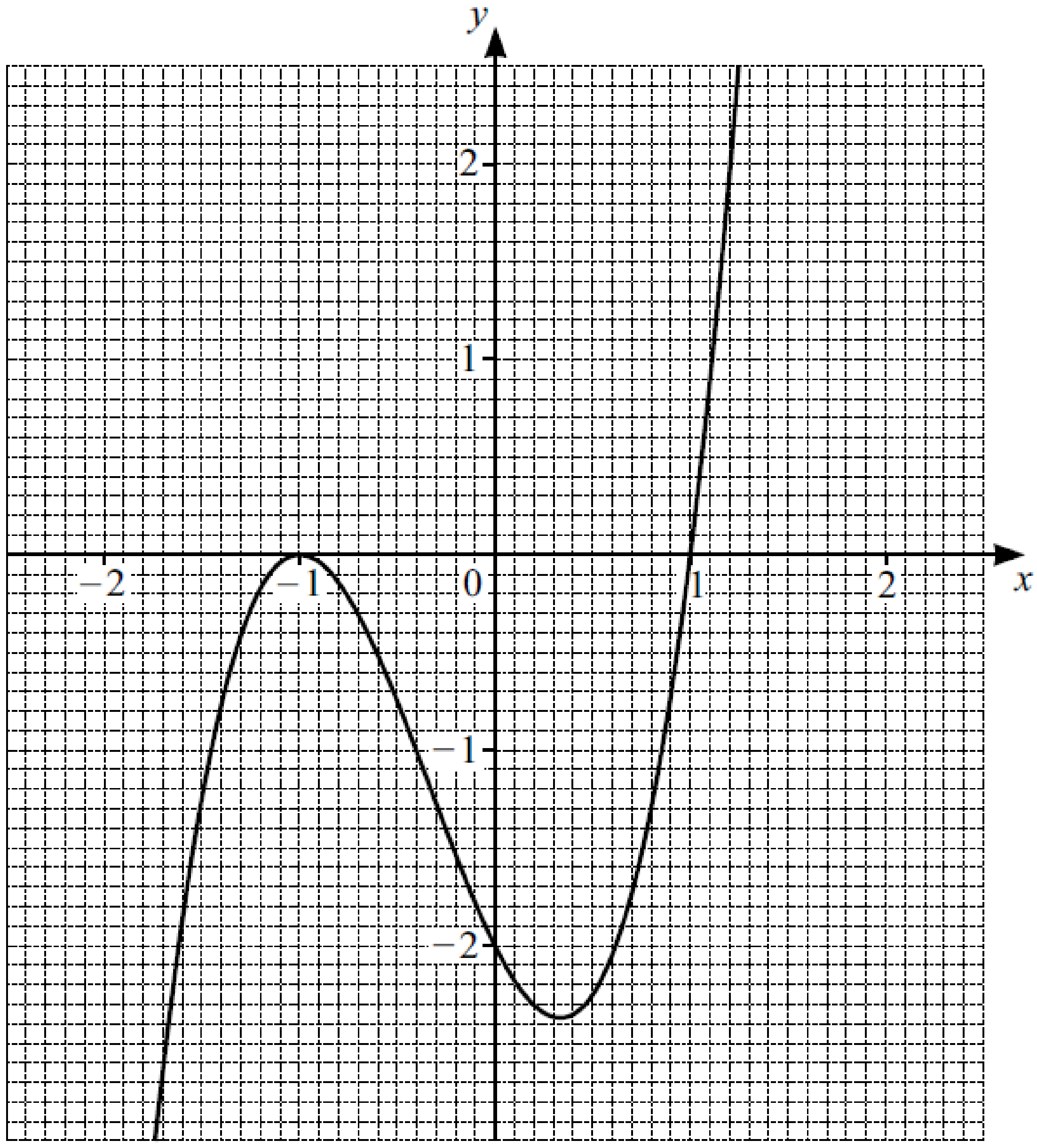
And solve

$$5x = -1$$

$x = -\frac{1}{5}$ [1]

(3 marks)

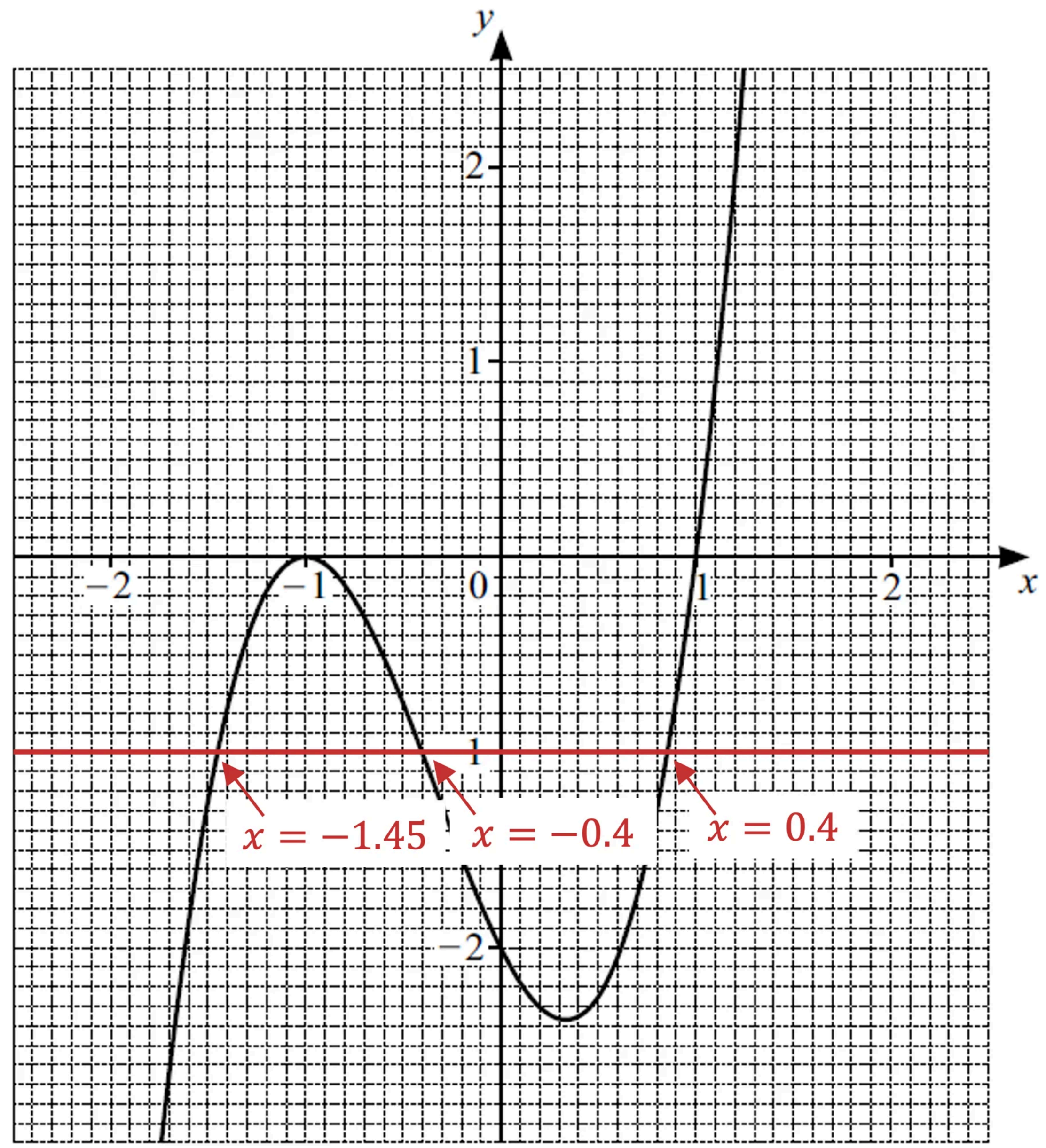
2



The diagram shows the graph of $y = f(x)$, where $f(x) = 2(x + 1)^2(x - 1)$.
Use the graph to solve the inequality $f(x) \leqslant -1$.

Answer

Find the critical values (the solutions of $f(x) = -1$) by drawing $y = -1$ on the diagram and reading the x values where it intersects $y = \mathrm{f}(x)$



$x = -1.45,\ x = -0.4,\ x = 0.85$

for the first and last value above, answers from -1.4 to 1.5 and from 0.8 to 0.9 are allowed [1]

Use these values to write as inequalities the regions of the diagram where the curve is below $y = -1$

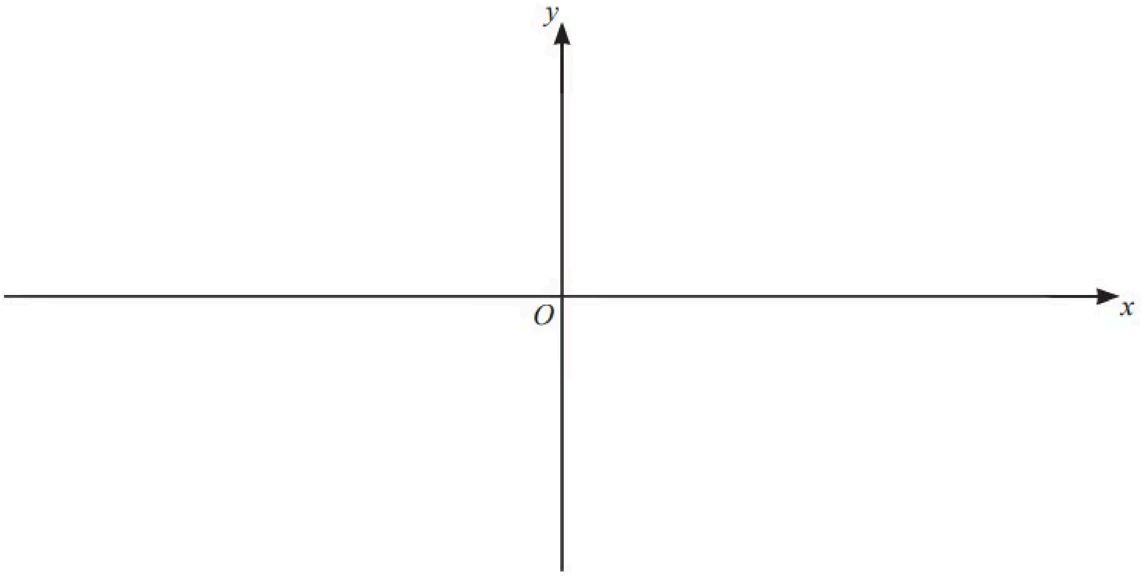
$x \leq 1.45$ and $-0.4 \leq x \leq 0.85$ [2]

one mark for each correct inequality

Take care to use "or equals to" inequalities, corresponding to the condition $f(x) \leq -1$

(3 marks)

3 (a) On the axes, sketch the graph of $y = 5(x+1)(3x-2)(x-2)$, stating the intercepts with the coordinate axes.



Answer

If we expand the brackets of the equation, the highest power of x we would get would be x^3. Therefore, the graph is cubic.

Solve $5(x+1)(3x-2)(x-2) = 0$ to find where the the curve will cross the x axis.

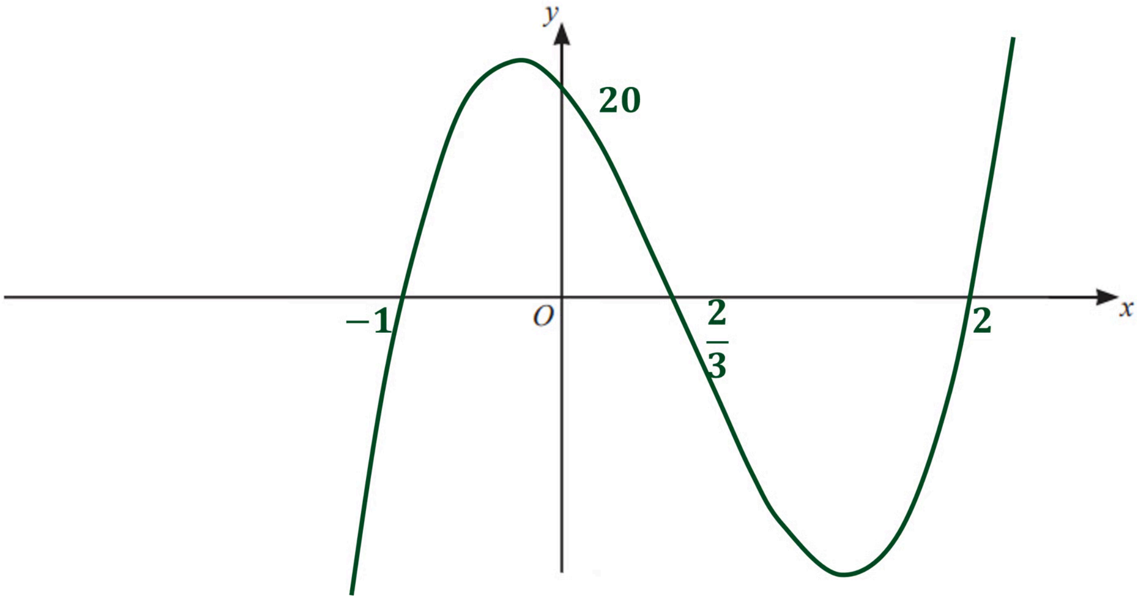
$$x = -1, \quad x = \frac{2}{3}, \quad x = 2$$

Substitute $x = 0$ into $y = 5(x+1)(3x-2)(x-2)$ to find where the curve will cross the y axis.

$$y = 5(1)(-2)(-2)$$

$$y = 20$$

Draw a cubic curve on the given axes. Label where the curve crosses the x and y axes.



Correct shape [1]

Correct roots at $x = -1, \quad x = 2, \quad x = \frac{2}{3}$ [1]

Correct y-intercept at $y = 20$ [1]

(3 marks)

(b) Hence find the values of x for which $5(x+1)(3x-2)(x-2) > 0$.

Answer

$y > 0$ above the x axis.

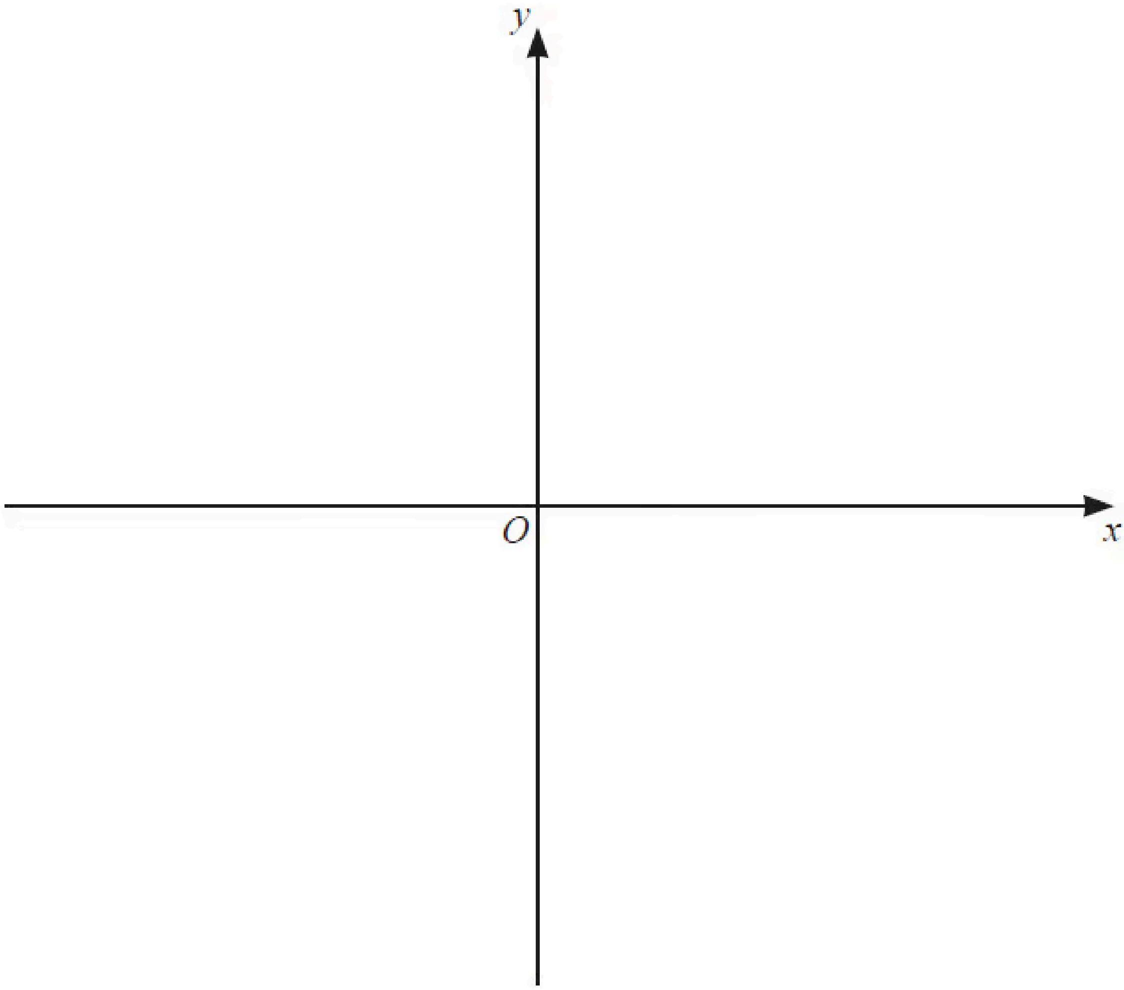
Look at the graph from part (a) to identify the x values where the curve is above the x axis.

$-1 < x < \frac{2}{3}$ [1]

$x > 2$ [1]

(2 marks)

4 (a) On the axes below sketch the graph of $y = -3(x-2)(x-4)(x+1)$, showing the coordinates of the points where the curve intersects the coordinate axes.



Answer

Notice that the graph will be a negative cubic since $-3 \times x \times x \times x = -3x^3$.

Find where the graph will cross the x-axis. This is true when $y = 0$.

$$0 = -3(x-2)(x-4)(x+1)$$

Either $x - 2 = 0$, $x - 4 = 0$ or $x + 1 = 0$

So $x = 2$, $x = 4$ or $x = -1$

Find where the graph will cross the y-axis. This is true when $x = 0$.

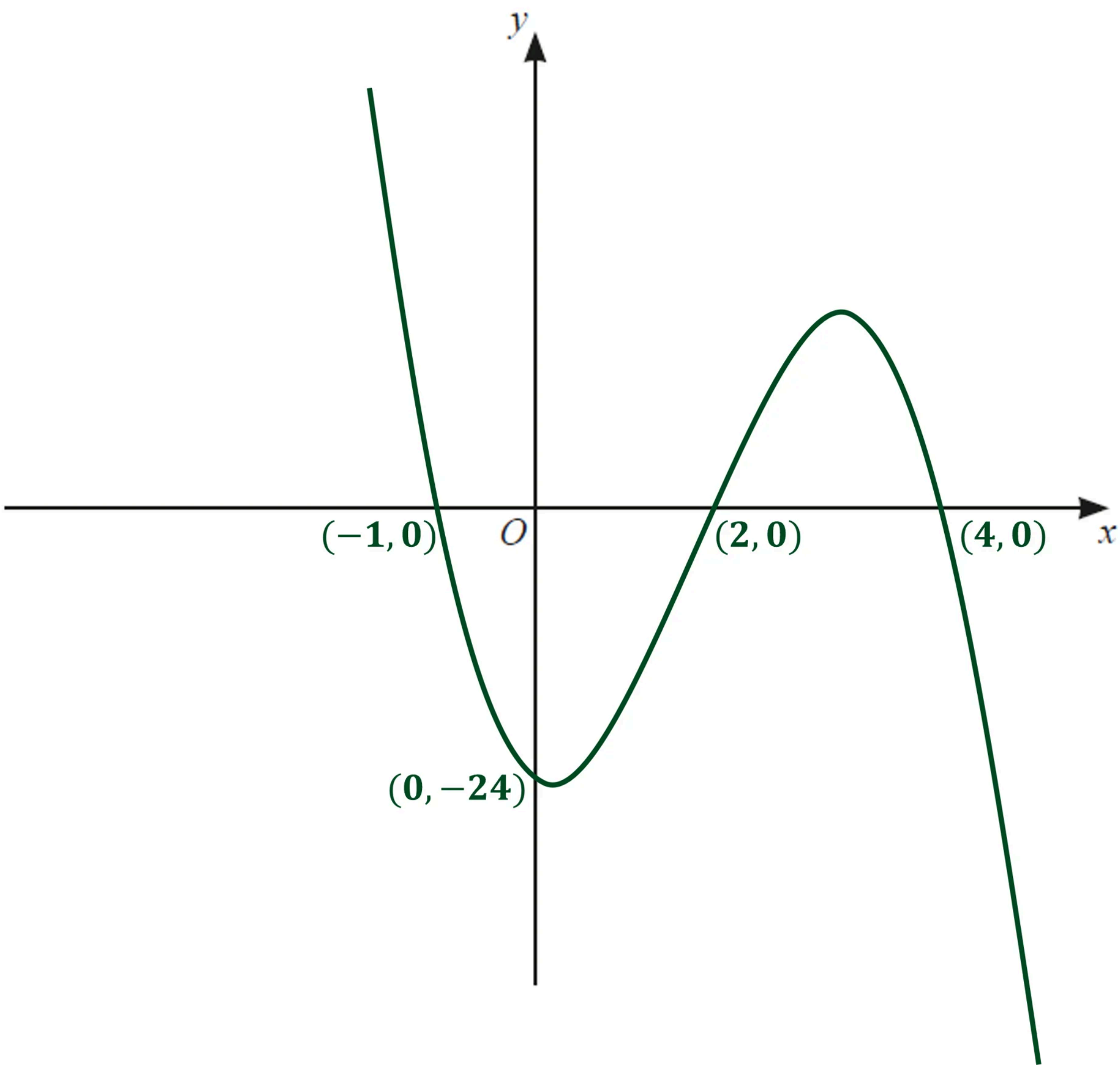
Substitute $x = 0$ into the equation giving:

$$y = -3(0-2)(0-4)(0+1)$$

$$y = -24$$

Plot the intercepts onto the graph which will be:

$$(2, 0), (4, 0), (-1, 0) \text{ and } (0, -24)$$



correct shape [1]
correct x-intercepts [1]
correct y-intercept [1]

(3 marks)

(b) Hence find the values of x for which $-3(x-2)(x-4)(x+1) > 0$.

Answer

We want to find where the graph is greater than 0 (but NOT equal to 0). In other words, where the graph is strictly above the x-axis.

The graph is above the x-axis in 2 sections:

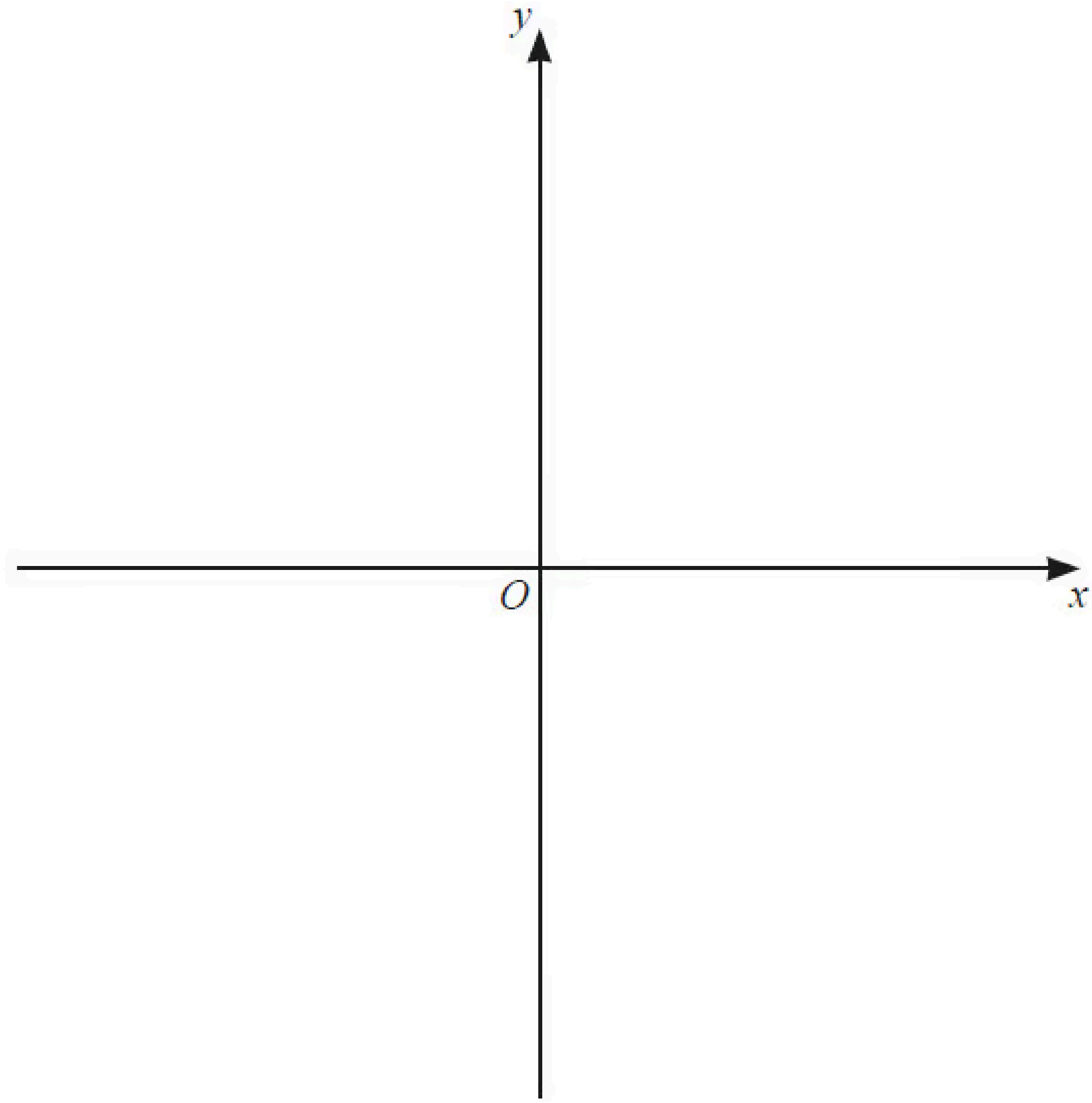
When $x < -1$ or when $2 < x < 4$

$x < -1$ or $2 < x < 4$

1 mark for each statement [1]

(2 marks)

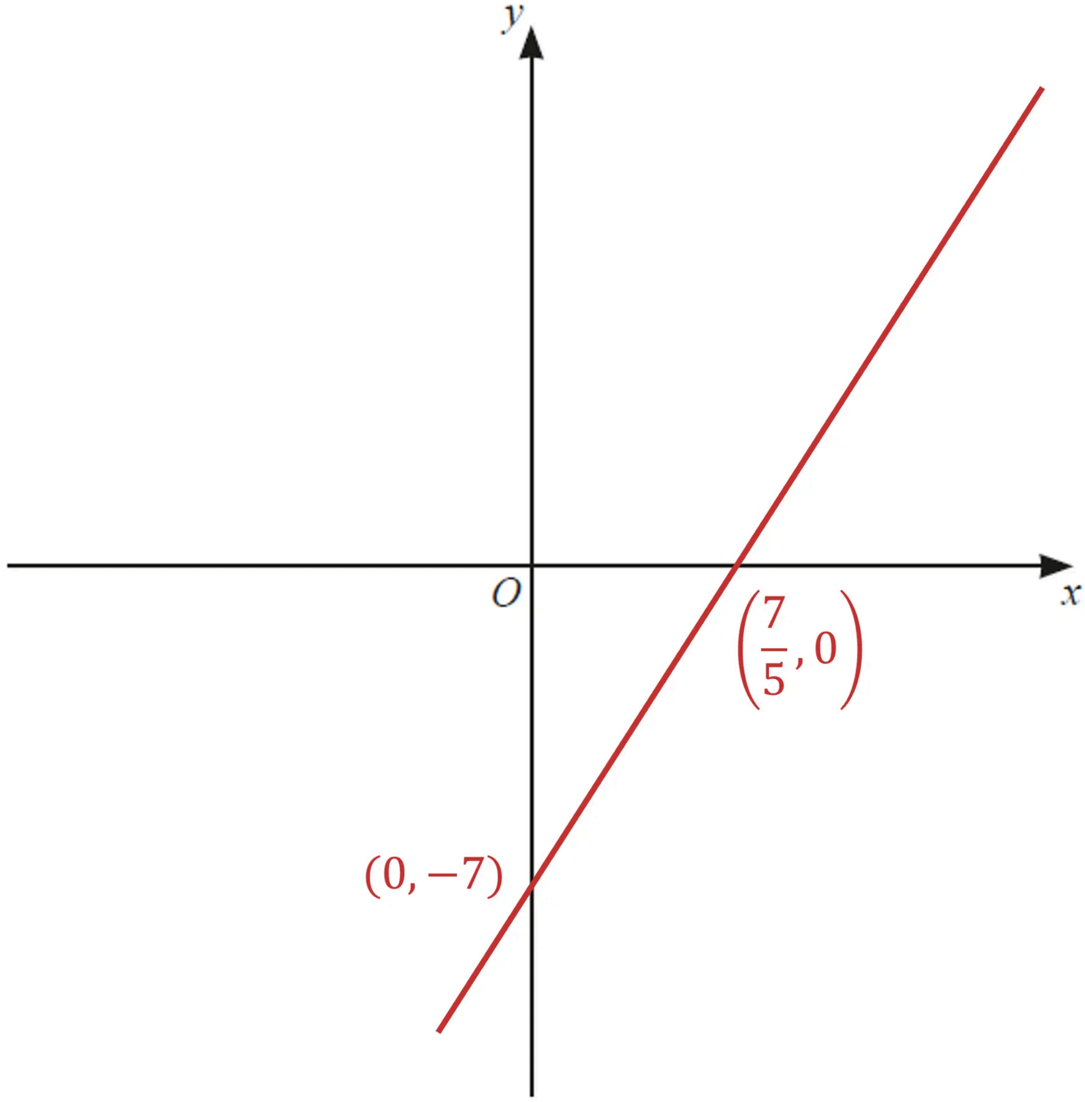
5 (a) On the axes below, sketch the graph of $y = |5x - 7|$, showing the coordinates of the points where the graph meets the coordinate axes.



Answer

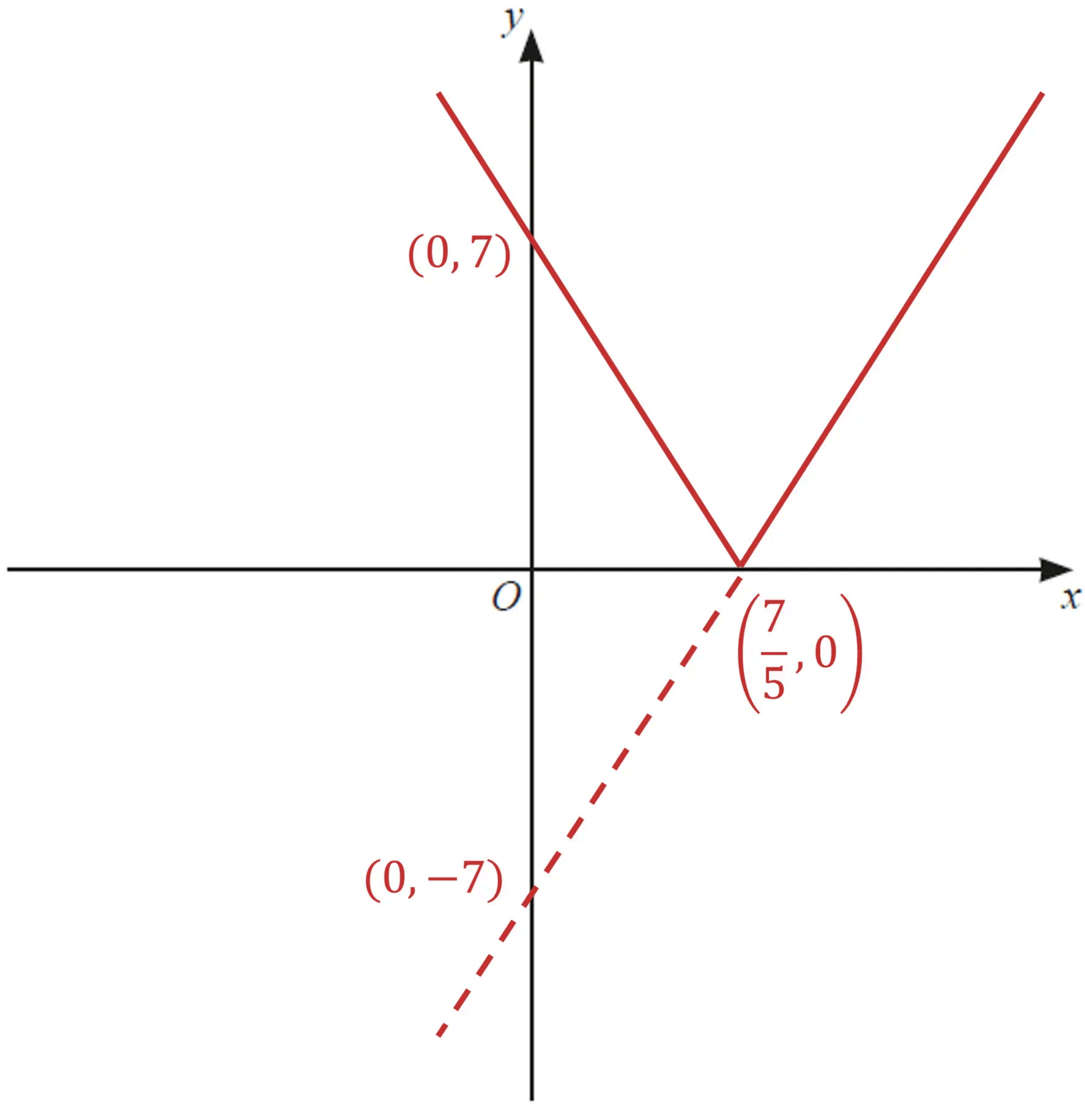
The line $y = 5x - 7$ would have y-intercept $(0, -7)$ and would intersect the x-axis at $\left(\frac{7}{5}, 0\right)$

Sketch this on the axes

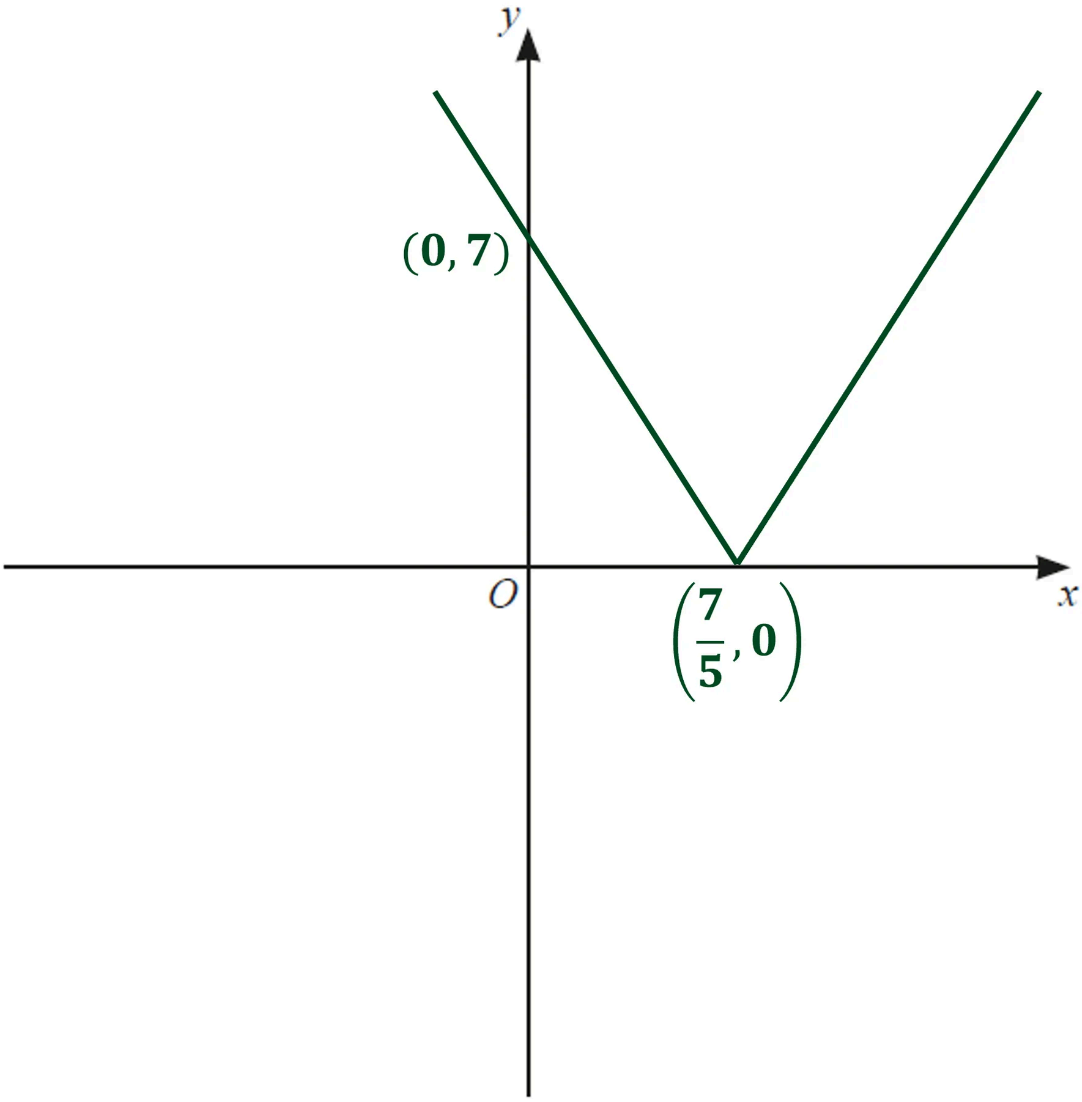


We need to sketch $y = |5x - 7|$ which means that all the y values are made positive. This means the part of the graph that goes below the x-axis gets reflected in the x-

axis:



The final graph should look like this



correct V shape with vertex on positive x-axis [1]

correct coordinate $(0, 7)$ [1]

correct coordinate $\left(\frac{7}{5}, 0\right)$ [1]

(3 marks)

(b) Solve $5|5x - 7| - 1 = 14$.

Answer

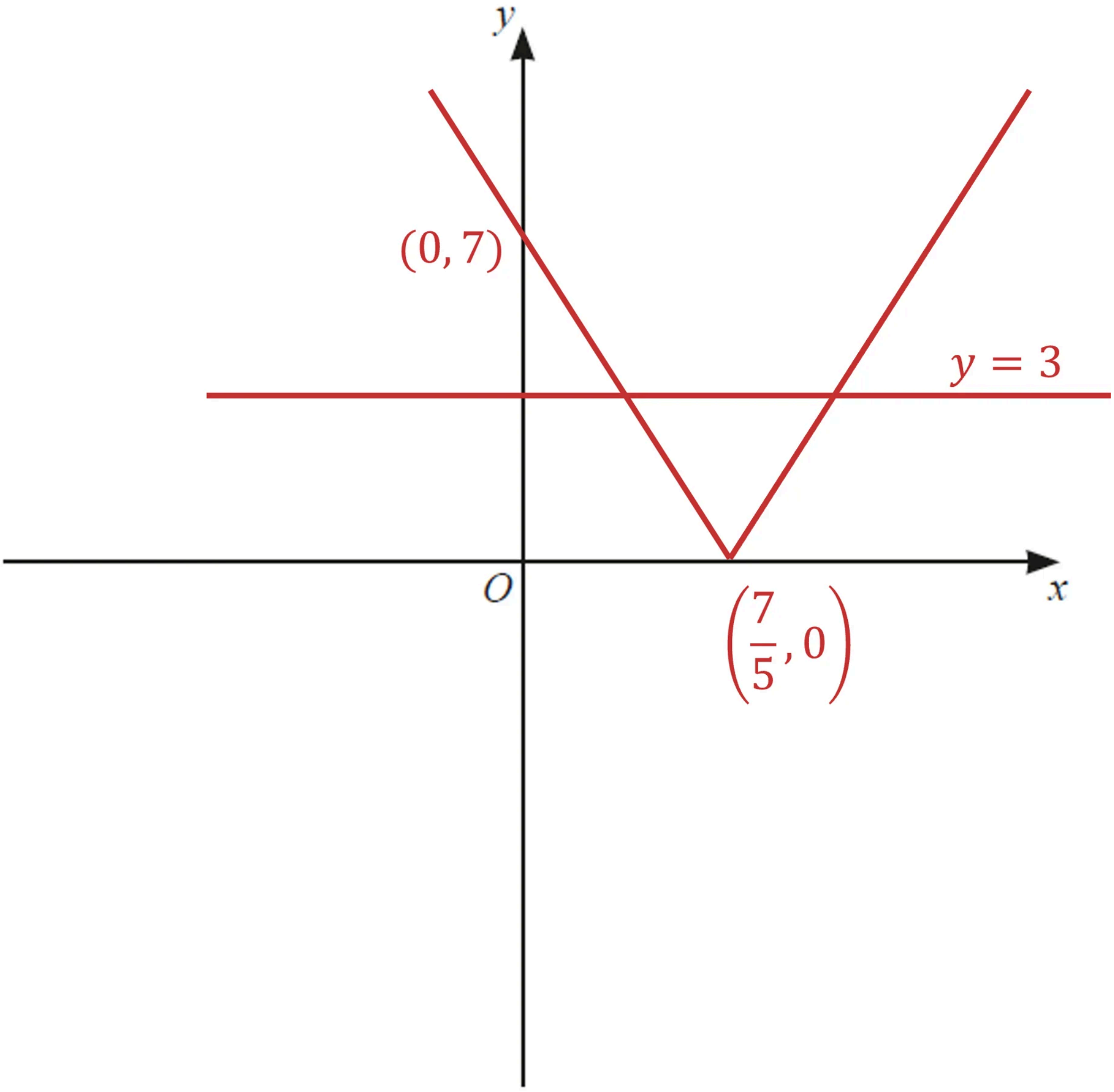
Add 1 to both sides and then divide by 5

$$5|5x-7|-1=14$$

$$5|5x-7|=15$$

$$|5x-7|=3$$

Use part (a) to see where the graph of $y=|5x-7|$ and the line $y=3$ intersect



There are 2 intersections so the original equation will have 2 solutions

Use the equation of the positive straight line and solve

$$5x-7=3$$

$$x = 2$$

[1]

Use the equation of the negative straight line and solve

$$-5x + 7 = 3$$

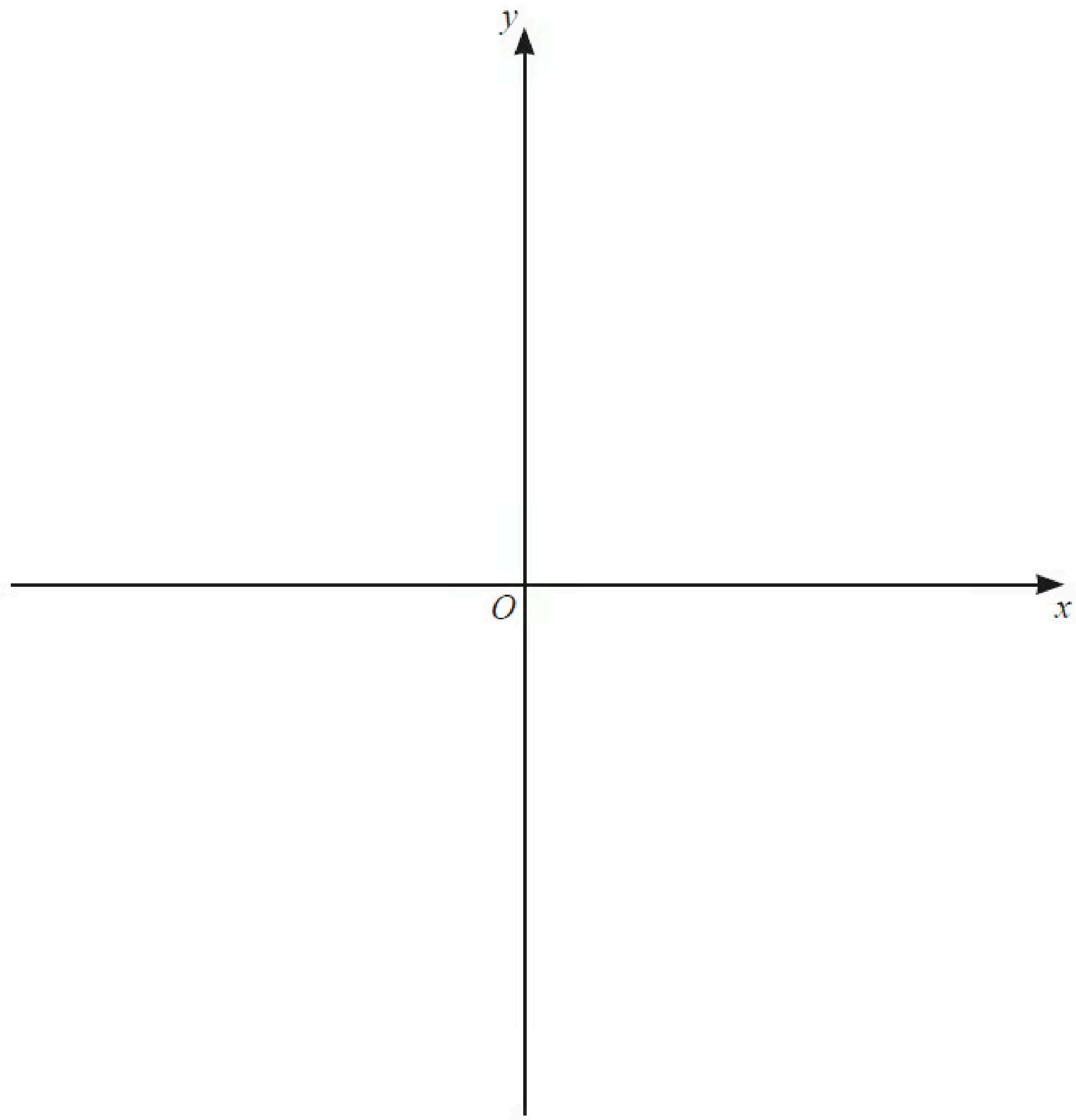
$$x = \frac{4}{5}$$

[1]

$x = 2$ or $x = \frac{4}{5}$ [1]

(3 marks)

6 (a) On the axes below, sketch the graph of $y = -(x+2)(x-1)(x-6)$, showing the coordinates of the points where the graph meets the coordinate axes.



Answer

Find where the graph intersects the x axis by substituting in $y = 0$

$$0 = -(x+2)(x-1)(x-6)$$

$$x = -2 \qquad x = 1 \qquad x = 6$$

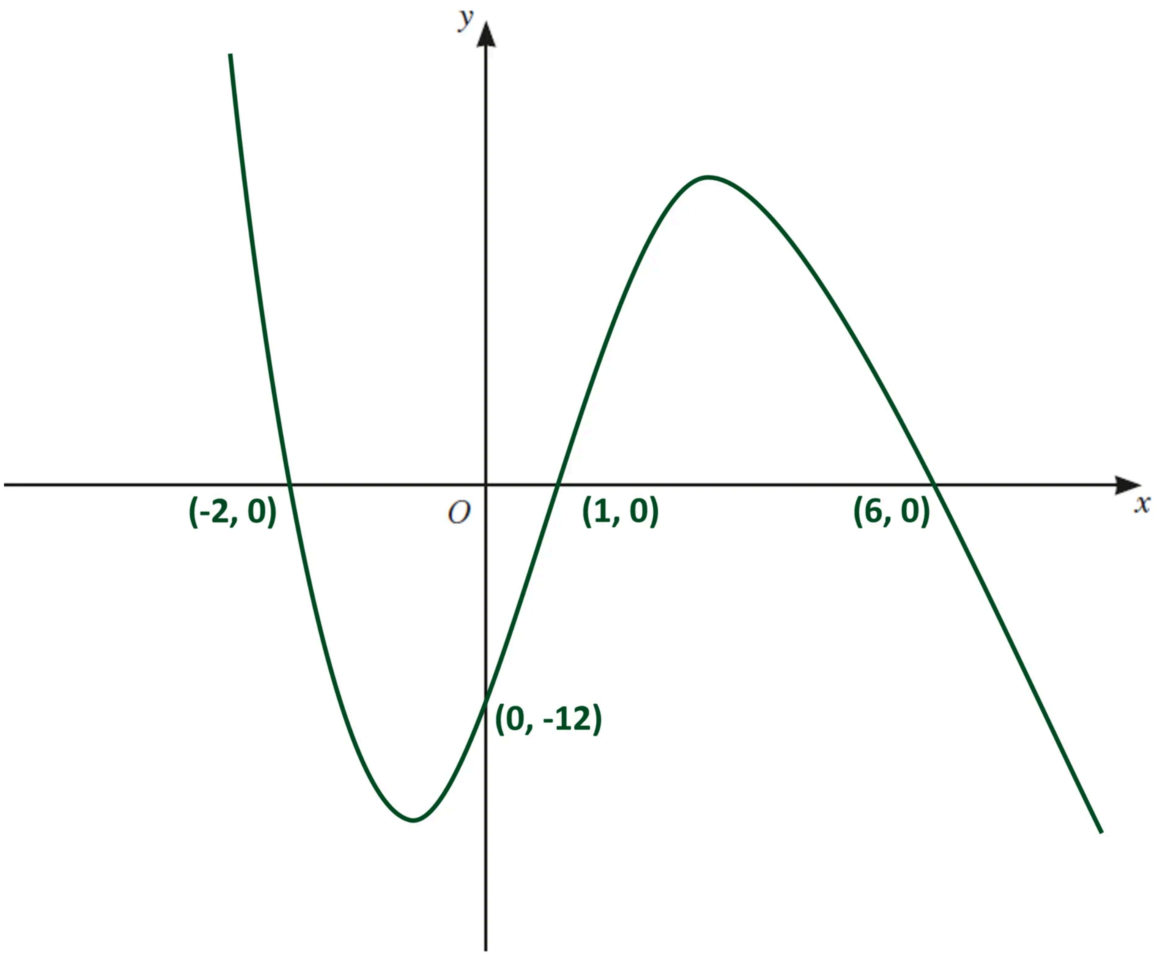
Find the y intercept by substituting in $x = 0$

$$y = -(2)(-1)(-6)$$

$$y = -12$$

The shape of the graph will be an inverted cubic.

Draw on the given axes, labeling where the curve crosses the axes.



correct shape [1]

correct coordinates $(-2, 0)$, $(1, 0)$, $(6, 0)$ *and* $(0, -12)$ [1]

(2 marks)

(b) Hence solve $-(x+2)(x-1)(x-6) \leq 0$.

Answer

$y \leq 0$ on and below the x axis.

Look at the graph from part (a) to identify the range of x values for which the curve is on or below the x axis.

$-2 \leq x \leq 1$ [1]

$x \geq 6$ [1]

(2 marks)

7 Solve the following equation.

$$2|x-1| \leq |2-x|$$

Answer

Method 1

Square both sides

$$4(x-1)^2 \leq (2-x)^2$$

[M1]

Expand and collect terms on one side

$$4(x^2-2x+1) \leq 4-4x+x^2$$
$$4x^2-8x+4 \leq 4-4x+x^2$$
$$3x^2-4x \leq 0$$

[A1]

Solve the quadratic inequality, e..g by factorising to find the critical points

$$x(3x-4) \leq 0$$

critical points are $x=0$ and $x=\frac{4}{3}$

[A1]

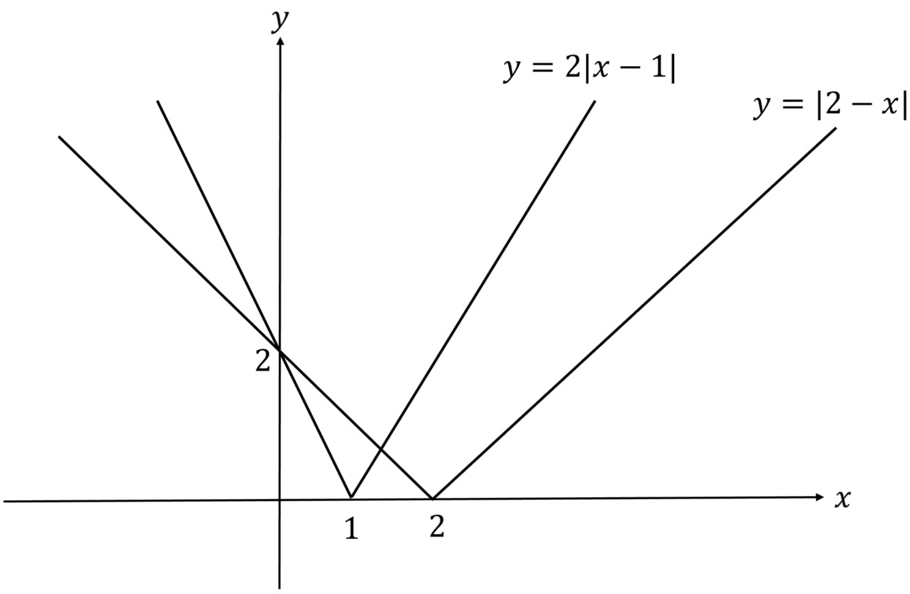
The curve $y = x(3x-4)$ is a positive quadratic (so goes below the x-axis between $x=0$ and $x=\frac{4}{3}$)

$0 \le x \le \frac{4}{3}$

[A1]

Method 2

Sketch $y=2|x-1|$ (graph 1) and $y=|2-x|$ (graph 2) on the same axes



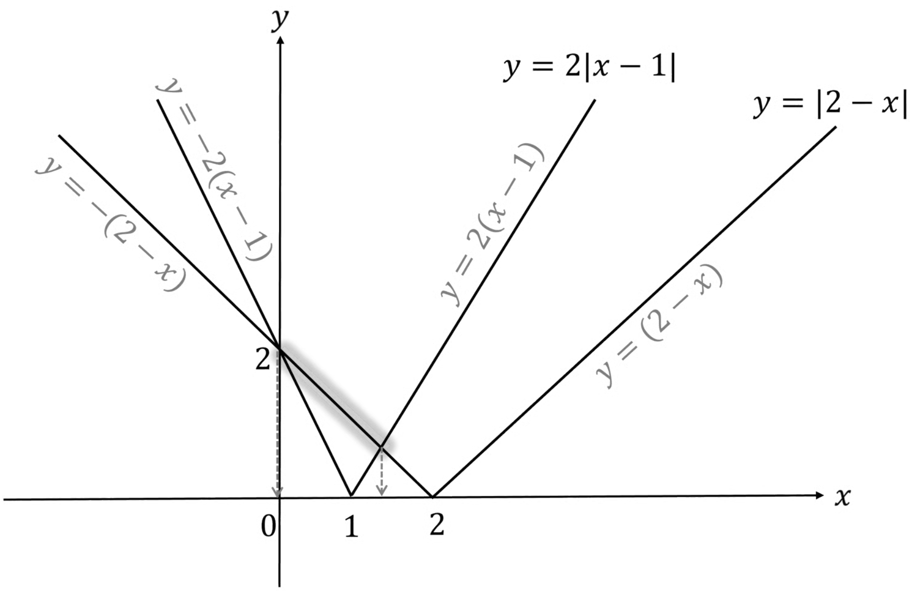
[B1 B1]

Mark Scheme and Guidance

B1: For at least sketching one graph correctly (numerical values not needed).

B1: For sketching both graphs correctly, intersecting each other on the positive y-axis (numerical values not needed).

Find where the line $y = 2|x-1|$ is always below the line $y = |2-x|$



Find the x-coordinates of this region (the first x-coordinate must be 0)

The second x-coordinate is the intersection of $y = 2(x-1)$ with $y = 2-x$

$$2(x-1) = 2-x$$

[M1]

Mark Scheme and Guidance

This mark is for identifying the two correct branches that intersect on the graph and using them to form an equation.

Solve

$$2(x-1) = 2-x$$
$$2x-2 = 2-x$$
$$3x = 4$$
$$x = \frac{4}{3}$$

Give the x-values for the region above as an inequality

$$0 \le x \le \frac{4}{3}$$

[A1]

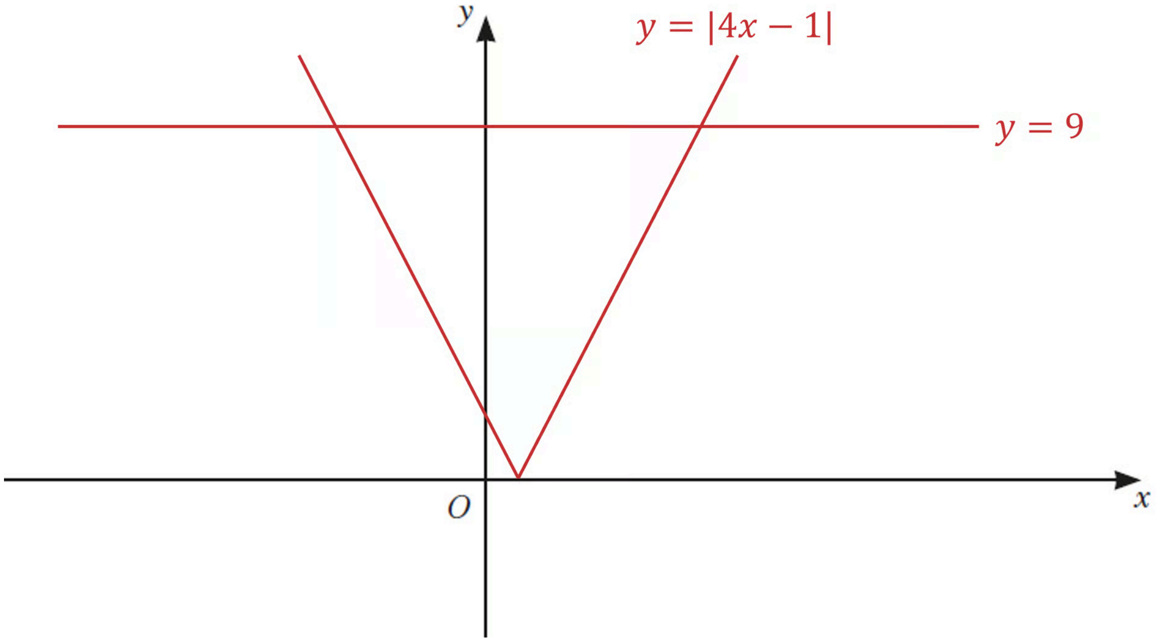
(4 marks)

Hard Questions

1 Solve the inequality $|4x-1| > 9$.

Answer

Sketch the two graphs.



To find the values of x when the two graphs are equation form two equtions.

$$4x-1=9$$
$$-(4x-1)=9$$

[1]

Solve each equation to find the x values.

$$4x-1=9 \Rightarrow x=\frac{5}{2}$$
$$1-4x=9 \Rightarrow x=-2$$

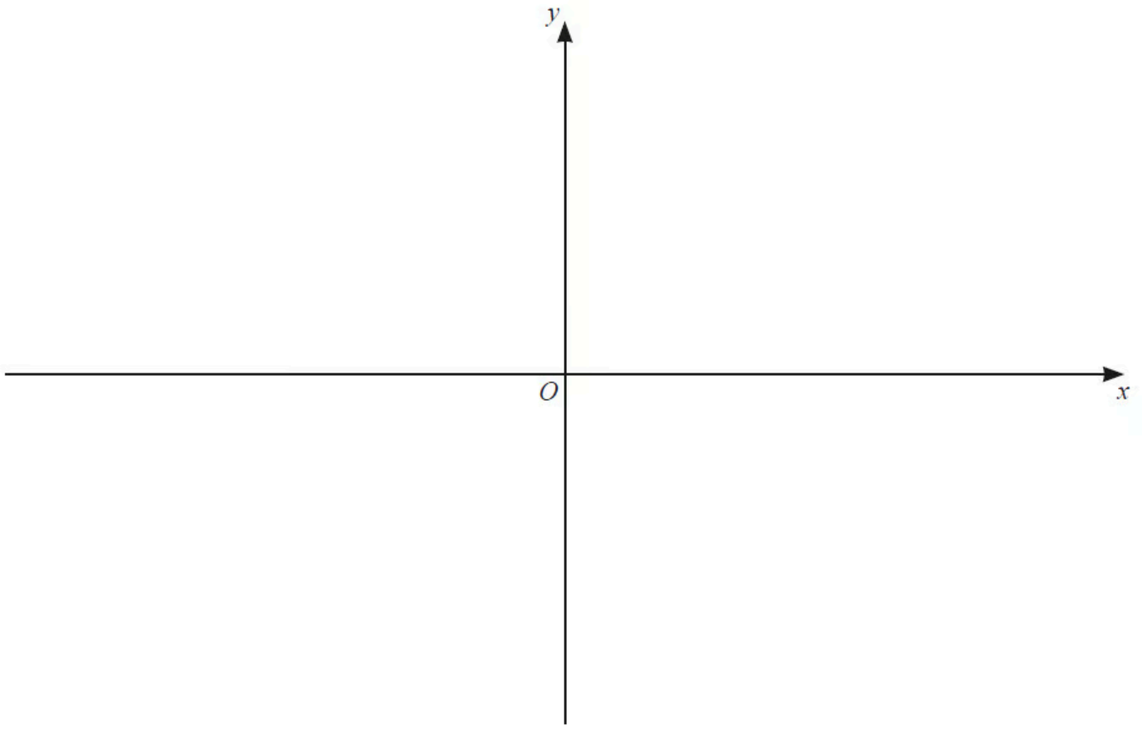
[1]

We want when the modulus function is greater than the constant function. This happens when the graph of the modulus function is above the graph of the straight line.

$x < -2$ or $x > \dfrac{5}{2}$ [1]

(3 marks)

2 On the axes below, sketch the graph of $y = |(x-2)(x+1)(x+2)|$ showing the coordinates of the points where the curve meets the axes.



Answer

Sketch the graph $y=(x-2)(x+1)(x+2)$ first

This will be a positive cubic graph since $x \times x \times x = x^3$

Find where the graph crosses the x-axis

When $y=0$,

$$0 = (x-2)(x+1)(x+2)$$

$$x = 2,\ -1 \text{ or } -2$$

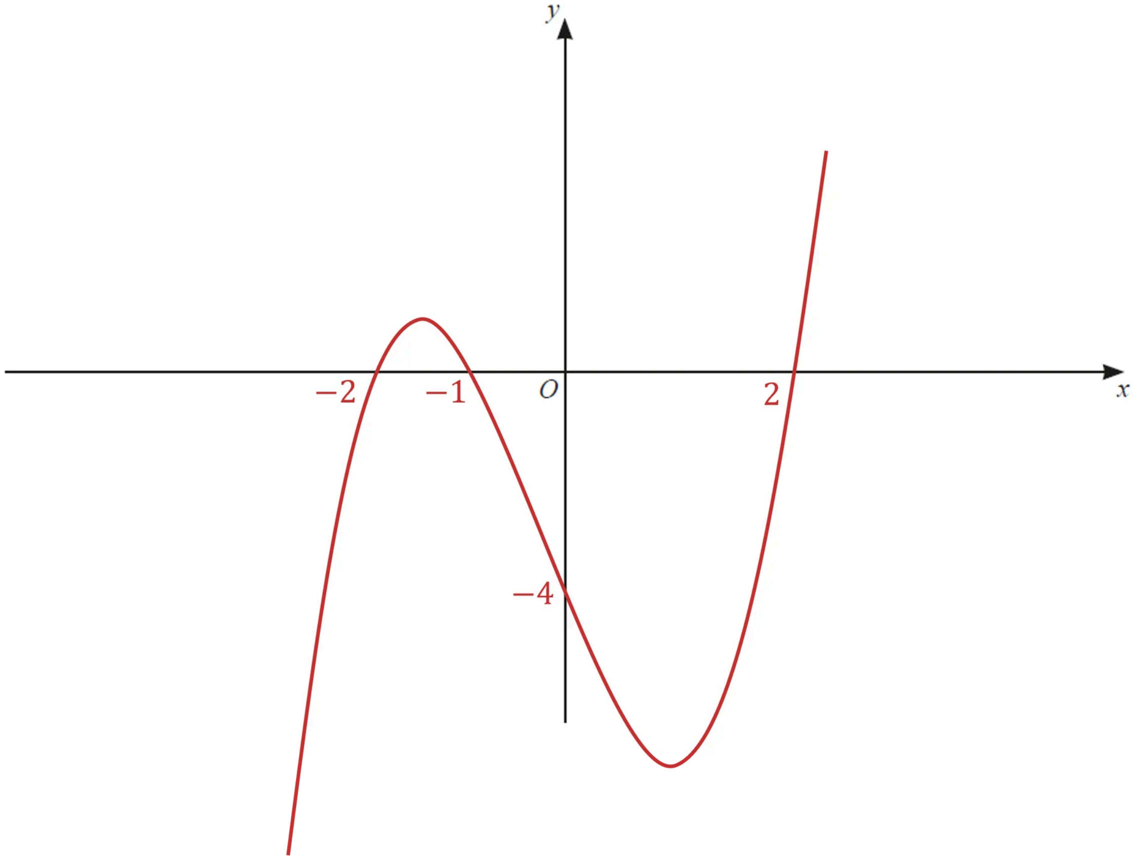
Find where the graph crosses the y-axis

When $x = 0$,

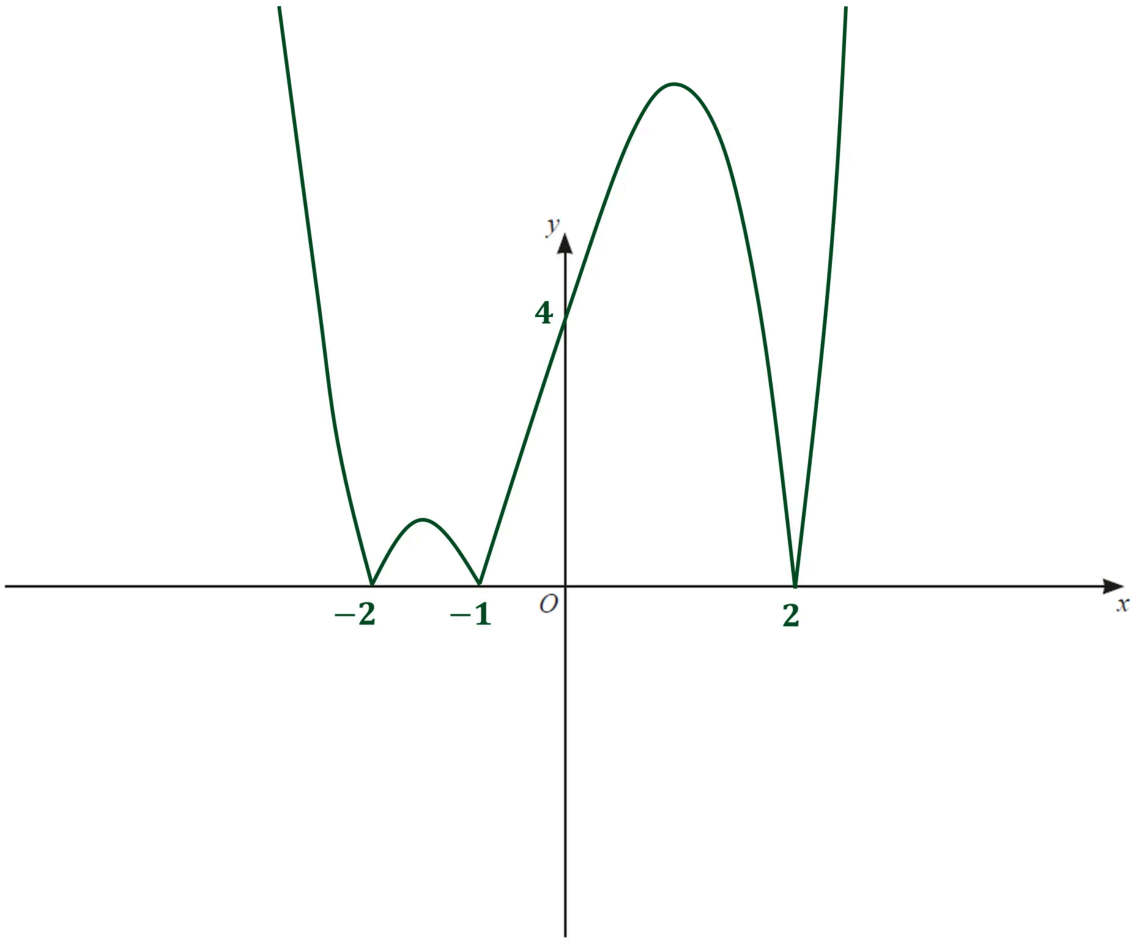
$$y = (0-2)(0+1)(0+2)$$

$$y = -4$$

Sketch the graph of $y = (x-2)(x+1)(x+2)$



The modulus graph $y = |(x-2)(x+1)(x+2)|$ will be similar to the graph above, but any parts below the x-axis will be reflected up above the x-axis



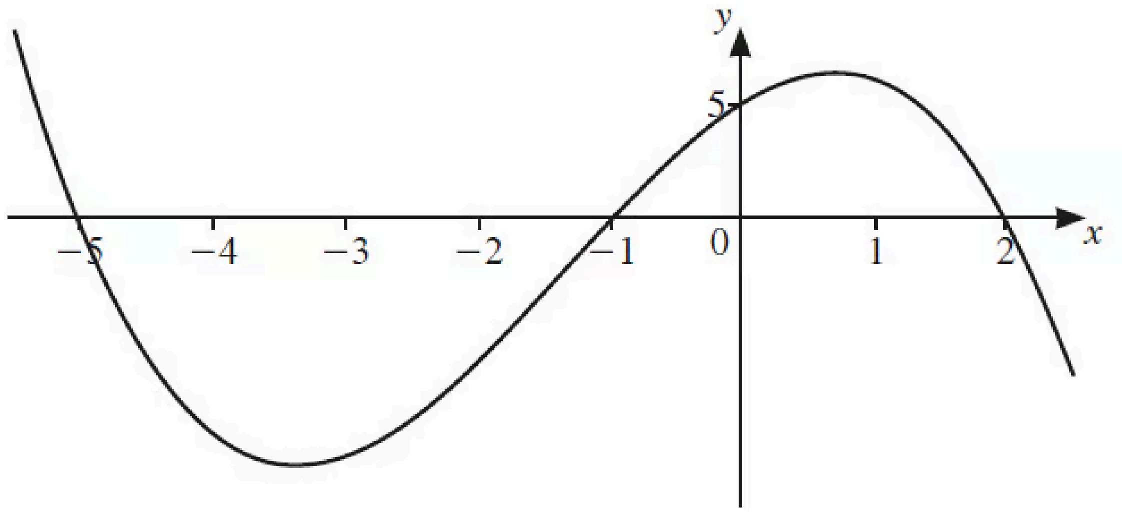
correct shape [1]

correct x-coordinates [1]

correct y-coordinate and max in first quadrant [1]

(3 marks)

3 (a)



The diagram shows the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a cubic polynomial. Find $\mathrm{f}(x)$.

Answer

The roots of the graph are $-5,\ -1$, and 2

Therefore, we can start with:

$$y = (x+5)(x+1)(x-2)$$

If I were to expand this, the constant at the end would be:

$$5 \times 1 \times -2 = -10$$

This would also be the y intercept of the graph.

Since we want a y intercept of 5, and an inverted cubic shape, we need to multiply our equation by $-\dfrac{1}{2}$

$$y = -\frac{1}{2}(x+5)(x+1)(x-2)$$

for negative [1]

for $\frac{1}{2}$ [1]

correct factors $(x+5)(x+1)(x-2)$ [1]

(3 marks)

(b) Write down the values of x such that $f(x) < 0$.

Answer

$f(x) < 0$ when the graph under the x axis.

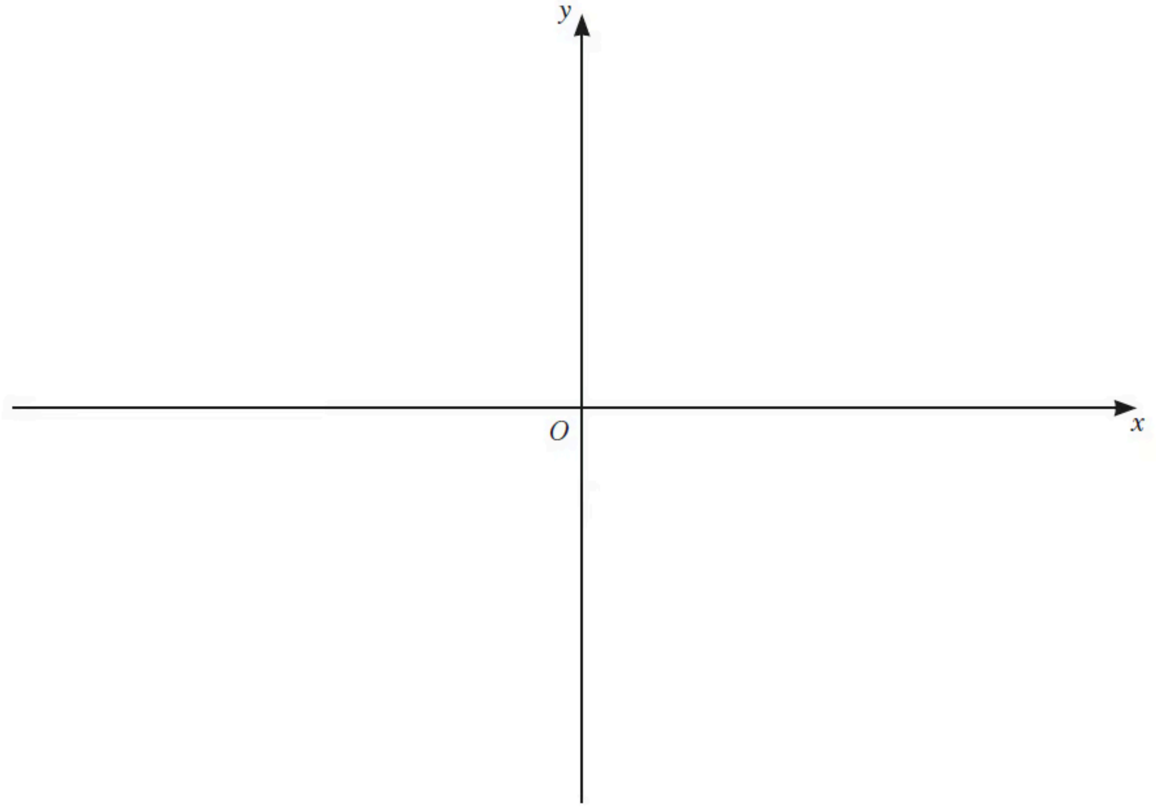
Examine the graph to decide between which values of x the graph is negative.

$x > 2$ [1]

$-5 < x < -1$ [1]

(2 marks)

4 (a) On the axes below, sketch the graph of $y = (x-2)(x+1)(3-x)$, stating the intercepts on the coordinate axes.



Answer

If we expand the brackets of the equation, the highest power of x we would get would be x^3. Therefore, the graph is cubic. Since one of the x in the brackets is negative, we will have an inverted cubic shape.

Solve $(x-2)(x+1)(3-x)=0$ to find where the the curve will cross the x axis.

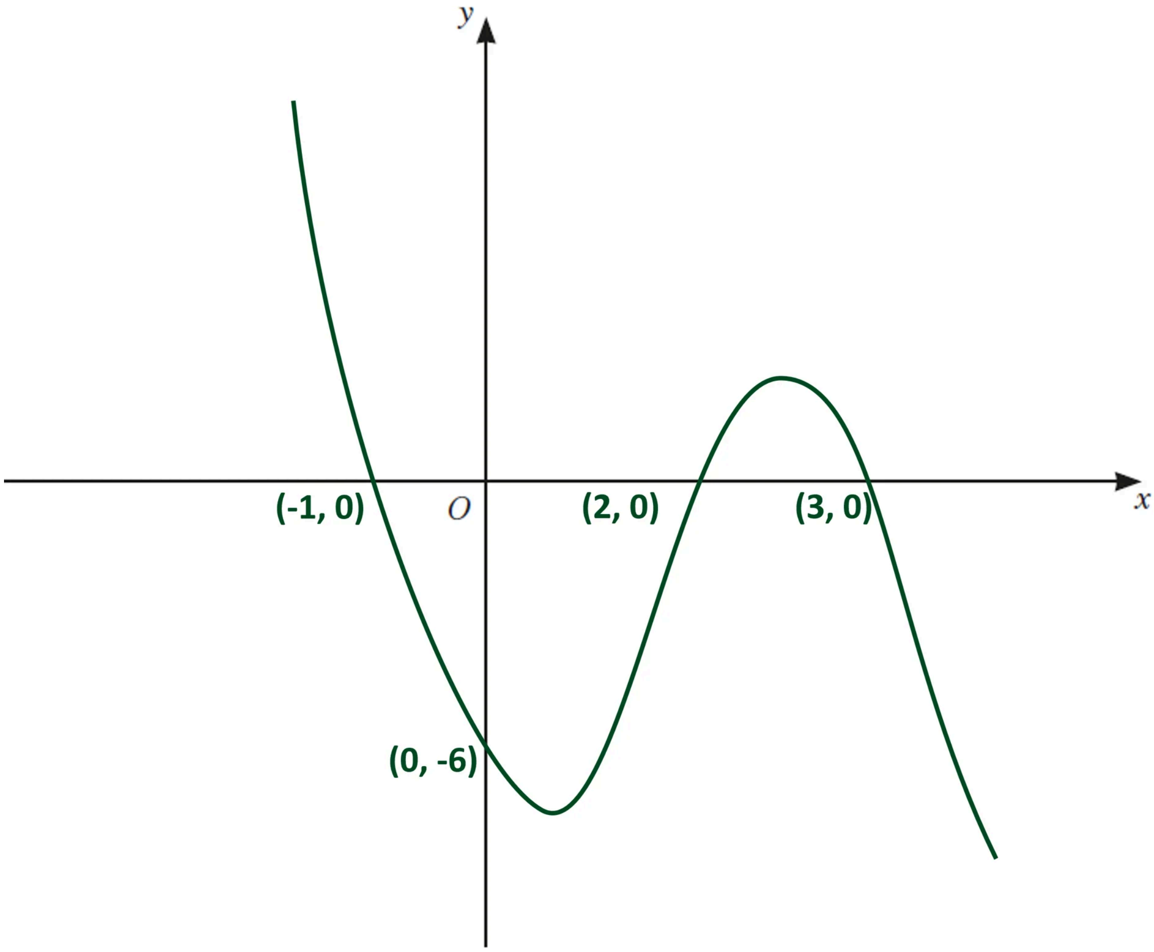
$$x=2, \quad x=-1, \quad x=3$$

Substitute $x=0$ into $y=(x-2)(x+1)(3-x)$ to find where the curve will cross the y axis.

$$y=(-2)(1)(3)$$

$$y=-6$$

Draw a cubic curve on the given axes. Label where the curve crosses the x and y axes.



correct shape [1]

correct x *axis intercepts* [1]

correct y *axis intercept* [1]

(3 marks)

(b) Hence write down the values of x such that $(x-2)(x+1)(3-x) > 0$.

Answer

$y > 0$ is above the x axis.

Look at the graph from part (a) to identify the x values where the curve is above the x axis.

$2 < x < 3$ [1]

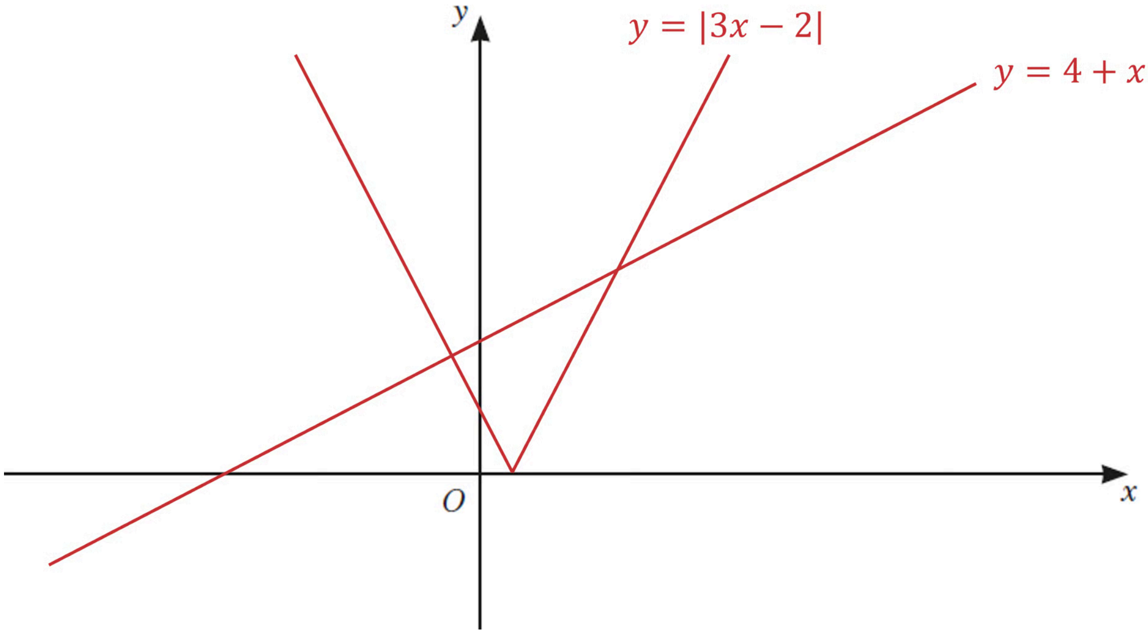
$x < -1$ [1]

(2 marks)

5 Solve $|3x-2| = 4+x$.

Answer

Sketch the lines $y=|3x-2|$ and $y=4+x$ on the same axes to identify how many intersections there are.



By examining the graph, we can see there is an intersection in the first quadrant. This is where $y=3x-2$ and $y=4+x$ cross.

$$3x-2=4+x$$
$$3x-x=4+2$$
$$2x=6$$
$$x=3$$

The other intersection is where the reflection of $y = 3x - 2$ in the x axis and the line $y = 4 + x$ cross.

The equation of this reflection is $y = -3x + 2$.

Therefore,

$$-3x + 2 = 4 + x$$

[1]

$$-3x - x = 4 - 2$$
$$-4x = 2$$
$$x = -0.5$$

$x = 3$ [1]

$x = -0.5$ [1]

(1 mark)

6 (a) The graph $y = \mathrm{p}(x)$ where $\mathrm{p}(x) = 5x^3 - 17x^2 + 16x + a$ intersects the x-axis at the points $(b, 0)$, $(1, 0)$ and $(c, 0)$ where $b < c$.

Find the constants a, b and c.

Answer

You are told that $(1, 0)$ is an x-intercept so substitute in $x = 1$ and $y = 0$ to find a

$$0 = 5(1)^3 - 17(1)^2 + 16(1) + a$$
$$0 = 5 - 17 + 16 + a$$
$$0 = 4 + a$$
$$a = -4$$

[B1]

Given that $x = 1$ is a solution to $\mathrm{p}(x) = 0$, the Factor theorem says that $(x - 1)$ is a factor

If $p(1)=0$ then $(x-1)$ is a factor of $p(x)=5x^3-17x^2+16x-4$

Use polynomial division to divide $p(x)$ by $(x-1)$ to find the quadratic factor

$$\begin{array}{r} 5x^2-12x+4 \\ (x-1)\overline{)5x^3-17x^2+16x-4} \\ \underline{5x^3-5x^2} \qquad\qquad\quad \\ -12x^2+16x \quad\;\; \\ \underline{-12x^2+12x} \quad\;\; \\ 4x-4 \\ \underline{4x-4} \\ 0 \end{array}$$

[M1]

$$5x^2-12x+4$$

[A1]

This means the cubic has the form

$$y=k(x-1)(5x^2-12x+4)$$

The x-intercepts are found when $y=0$

Solve $5x^2-12x+4=0$ to find the remaining two x-intercepts

$$(5x-2)(x-2)$$

[M1]

Mark Scheme and Guidance

This mark is for using a correct method to solve your 3-term quadratic factor equal to zero.

Recall that the question labels them $(b, 0)$, $(1, 0)$ and $(c, 0)$ where $b<c$

$$b = \frac{2}{5} \text{ and } c = 2$$

[A1]

Present all your answers from above

$a = -4$, $b = \frac{2}{5}$ **and** $c = 2$

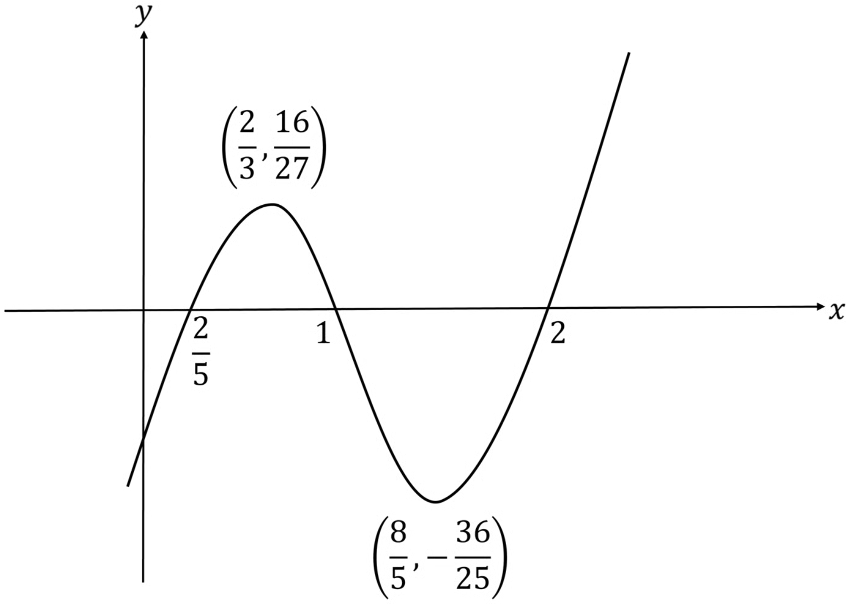
(5 marks)

(b) The curve $y = \mathrm{p}(x)$ has a local maximum point at $\left(\frac{2}{3}, \frac{16}{27}\right)$ and a local minimum point at $\left(\frac{8}{5}, -\frac{36}{25}\right)$.

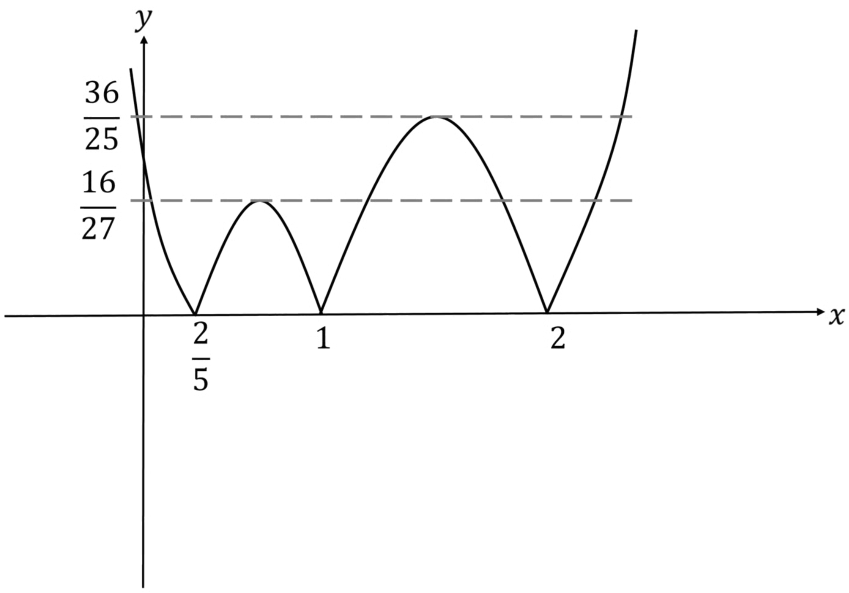
Find the value of d for which the equation $|p(x)| = d$ has exactly 5 distinct solutions.

Answer

It helps to imagine the cubic $y = \mathrm{p}(x)$ with the information given



Then imagine the modulus of this cubic equation, $y = |\mathrm{p}(x)|$



[M1]

Mark Scheme and Guidance

This mark is for attempting to sketch $y = |\mathrm{p}(x)|$ with the peaks in the correct size order (no numerical values are needed for this mark)

You need to find the horizontal line, $y = d$, that intersects the graph $y = |\mathrm{p}(x)|$ exactly 5 times

This only happens with the line $y = \dfrac{16}{27}$

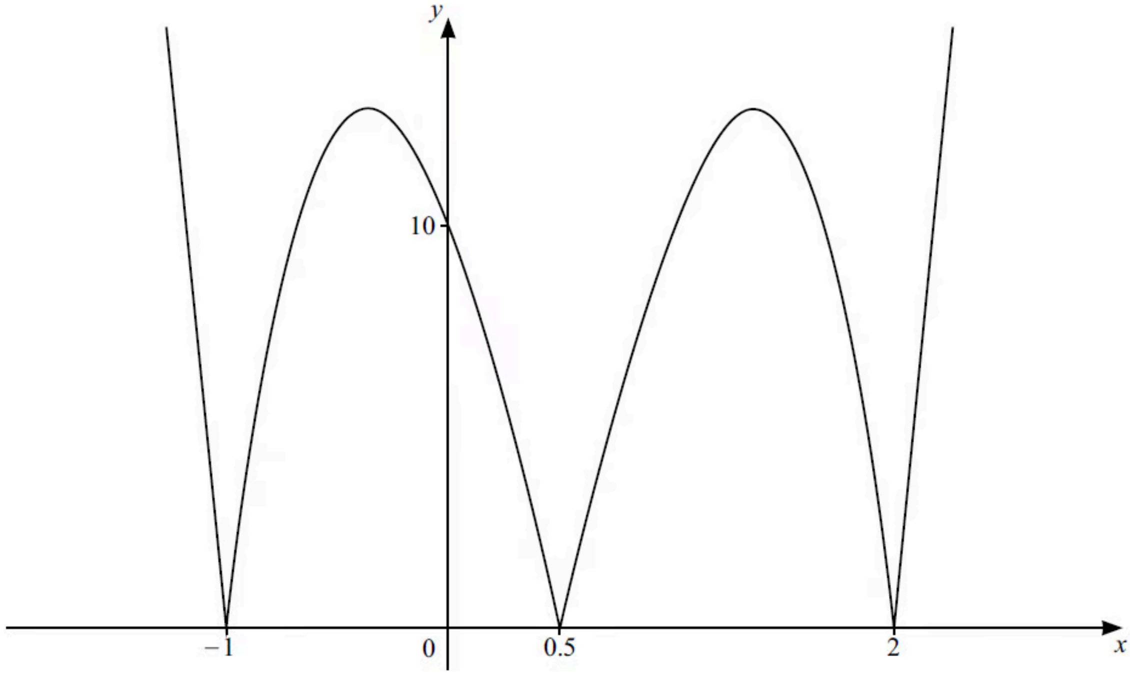
$d = \dfrac{16}{27}$

[A1]

(2 marks)

Very Hard Questions

1



The diagram shows the graph of $y = |f(x)|$, where $f(x)$ is a cubic function. Find the possible expressions for $f(x)$ in factorised form.

Answer

The function has roots at $x = -1$, $x = \frac{1}{2}$ and $x = 2$, so due to the factor theorem we know that:

$$f(x) = a(x+1)(2x-1)(x-2)$$

'$(x+1)(2x-1)(x-2)$' *or an equivalent factorisation* [1]

The *y*–intercept of the function is (0, 10) so substitute $x = 0$ and $f(x) = 10$ into the above, and solve to find *a*.

$$10 = a(0+1)(2(0)-1)(0-2)$$
$$10 = a(1)(-1)(-2)$$
$$10 = 2a$$
$$a = 5$$

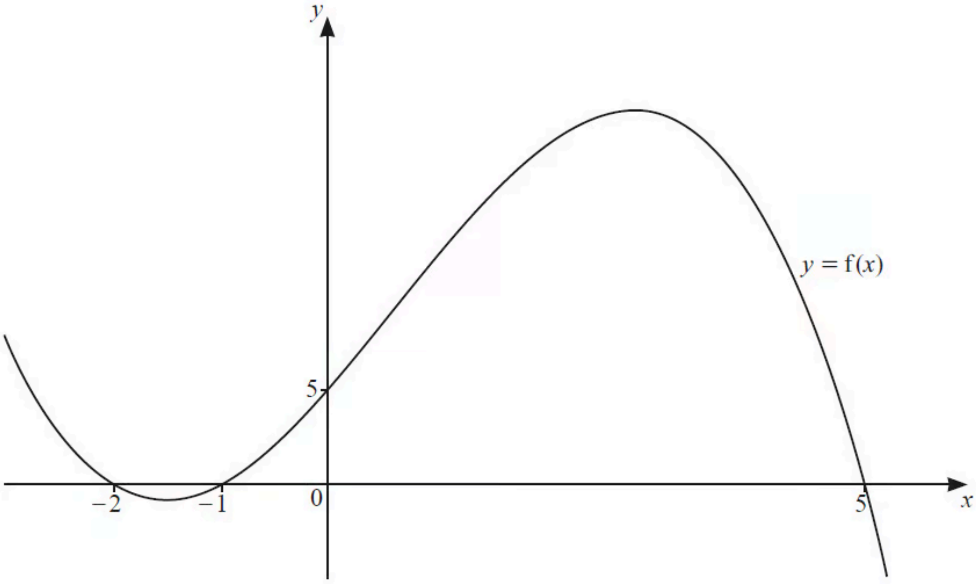
[1]

Remember that the graph is of the modulus of $\text{f}(x)$ so a could be positive or negative 5

$$\text{f}(x) = \pm 5(x+1)(2x-1)(x-2)$$ [1]

(3 marks)

2 (a) The diagram shows the graph of a cubic curve .



Find an expression for $\text{f}(x)$.

Answer

The graph crosses the *x* axis at the roots of the equation, and this will help us to identify the factors:

$$x = -2, x = -1, x = 5$$

$$\text{f}(x) = a(x+2)(x+1)(x-5)$$

correct factors [1]

This would give us a y-intercept of -10, but the graph shows us that the y-intercept is 5, so we need to adjust our expression with a reflection in the *x* axis and a vertical stretch of $-\dfrac{1}{2}$

$$f(x) = -\frac{1}{2}(x + 2)(x + 1)(x - 5)$$

$$f(x) = -\frac{1}{2}(x + 2)(x + 1)(x - 5)$$ [1]

(2 marks)

(b) Solve $f(x) \leq 0$.

Answer

We are looking for the parts of the graph that are below the x-axis. This happens when:

$$-2 \leq x \leq -1$$

and

$$x \geq 5$$

$$-2 \leq x \leq -1, \quad x \geq 5$$

1 mark for each statement [2]

(2 marks)

3 (a) Write $2x^2 + 3x - 4$ in the form $a(x+b)^2 + c$, where a, b and c are constants.

Answer

Factorise the first two terms to make the coefficient of x^2 equal 1 inside the brackets.

$$2\left[x^2 + \frac{3}{2}x\right] - 4$$

[1]

Complete the square for the expression in the brackets. To complete the square,
$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

$$2\left[\left(x + \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right] - 4$$

[1]

Simplify.

$$2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}\right] - 4$$

Expand the outer brackets only and simplify.

$$2\left(x + \frac{3}{4}\right)^2 - (2)\left(\frac{9}{16}\right) - 4$$

$$2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}$$ [1]

(3 marks)

(b) Hence write down the coordinates of the stationary point on the curve $y = 2x^2 + 3x - 4$.

Answer

The minimum point on the curve $y = (x+a)^2 + b$ is $(-a, b)$.

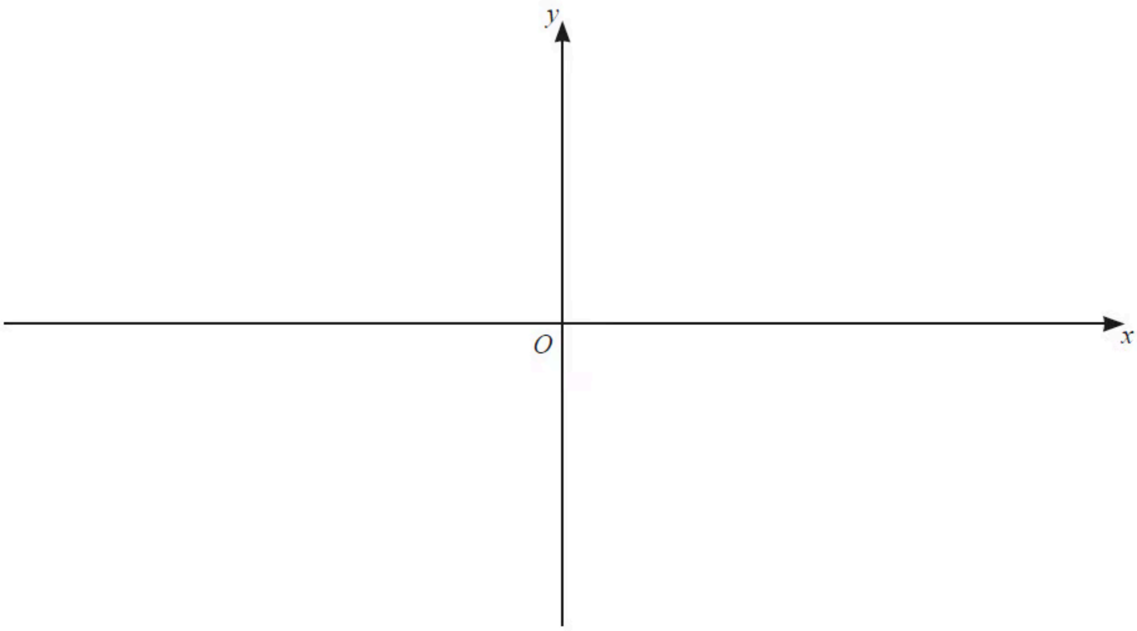
For $y = \left(x + \frac{3}{4}\right)^2 - \frac{41}{8}$,

$$\left(-\frac{3}{4}, -\frac{41}{8}\right)$$

1 mark for each coordinate [2]

(2 marks)

(c) On the axes below, sketch the graph of $y = |2x^2 + 3x - 4|$, showing the exact values of the intercepts of the curve with the coordinate axes.



Answer

Solve $y = 2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}$ when $y = 0$ to find the x intercepts.

$$2\left(x+\frac{3}{4}\right)^2-\frac{41}{8}=0$$

$$2\left(x+\frac{3}{4}\right)^2=\frac{41}{8}$$

$$\left(x+\frac{3}{4}\right)^2=\frac{41}{16}$$

$$x+\frac{3}{4}=\pm\sqrt{\frac{41}{16}}$$

$$x=\frac{\pm\sqrt{41}}{4}-\frac{3}{4}$$

$$x=\frac{-3\pm\sqrt{41}}{4}$$

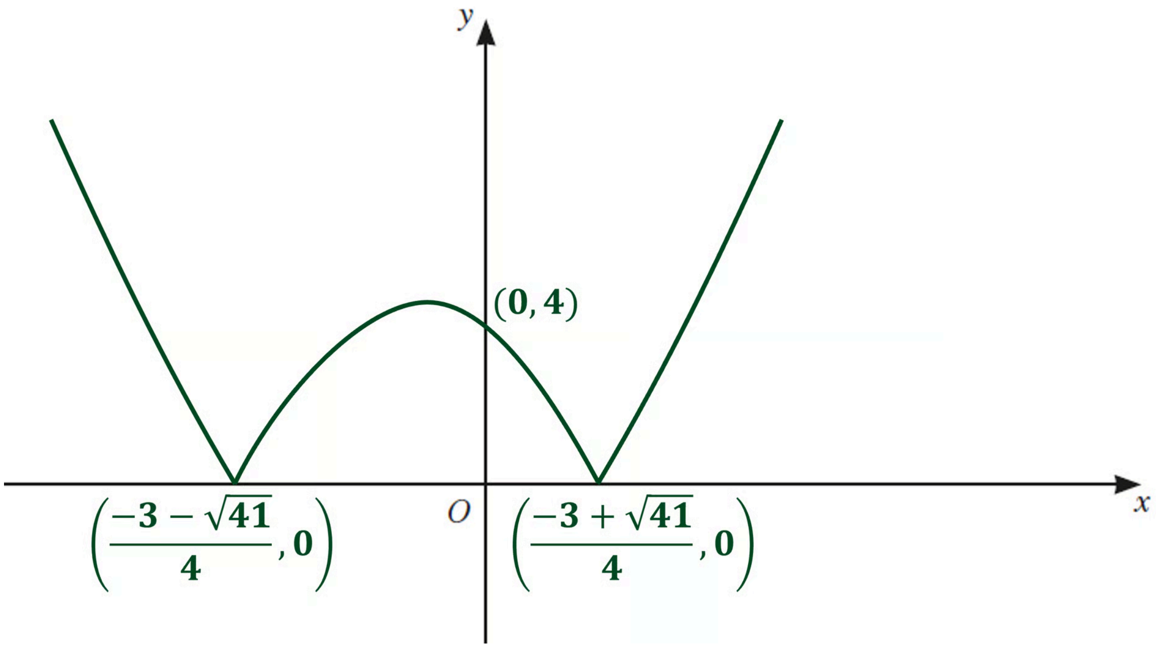
Solve $y=|2x^2+3x-4|$ when $x=0$ to find the $y$intercept.

$$y=|2(0)^2+3(0)-4|$$

$$y=|-4|$$

$$y=4$$

$y = 2x^2 + 3x - 4$ is a parabola. Since we are taking the modulus, everything below the x axis will be reflected above the x axis.



correct shape [1]

correct x intercepts $\frac{-3 \pm \sqrt{41}}{4}$ [1]

correct y intercept of 4 [1]

(3 marks)

(d) Find the value of k for which $|2x^2 + 3x - 4| = k$ has exactly 3 values of x.

Answer

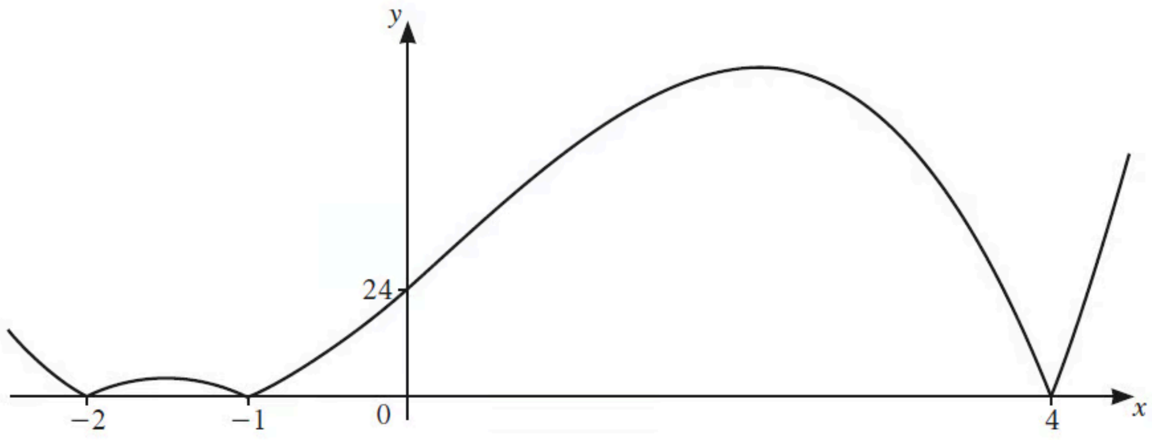
Using the graph from part c, identify where we would need to draw the horizontal line $y = k$ to achieve three intersections with the curve.

Three intersections are achieved at the reflection of the minimum point found in part b, when the line is $y = \frac{41}{8}$

$\frac{41}{8}$ [1]

(1 mark)

4



The diagram shows the graph of $y = |p(x)|$, where $p(x)$ is a cubic function. Find the two possible expressions for $p(x)$.

Answer

The graph crosses the x-axis at the roots of the equation, and this will help us to identify the factors:

$$x = -2, x = -1. x = 4$$

This means that the brackets must be

$$f(x) = (x + 2)(x + 1)(x - 4)$$

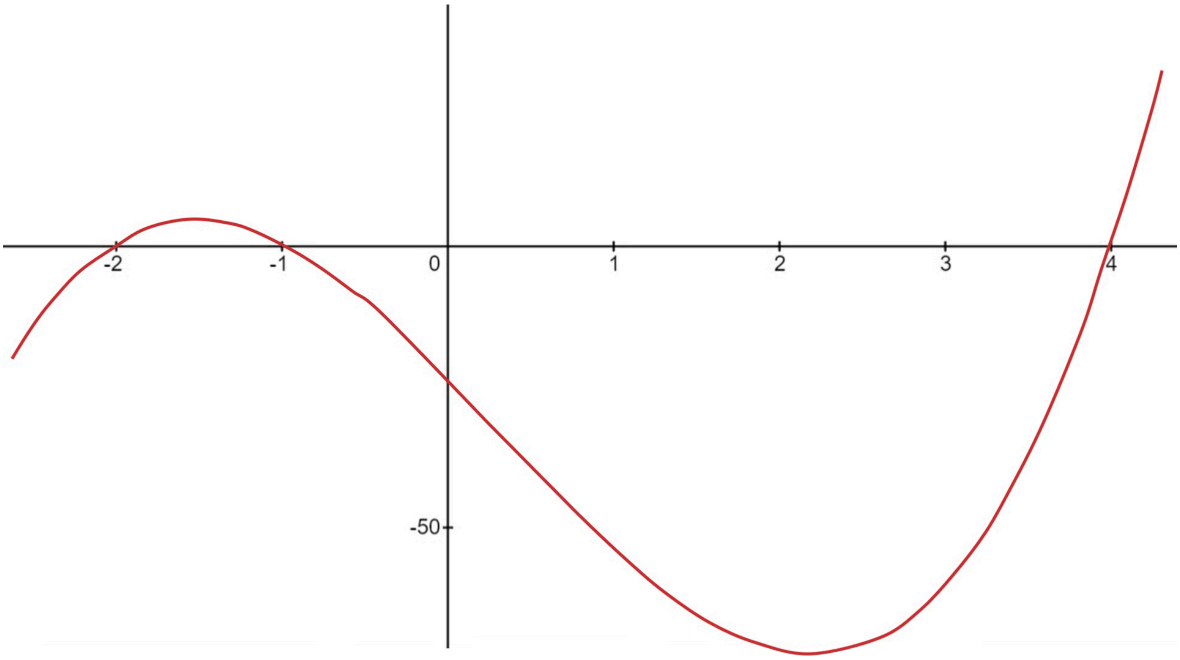
correct brackets [1]

If we expand the brackets, we get

$$f(x) = x^3 - x^2 - 10x - 8$$

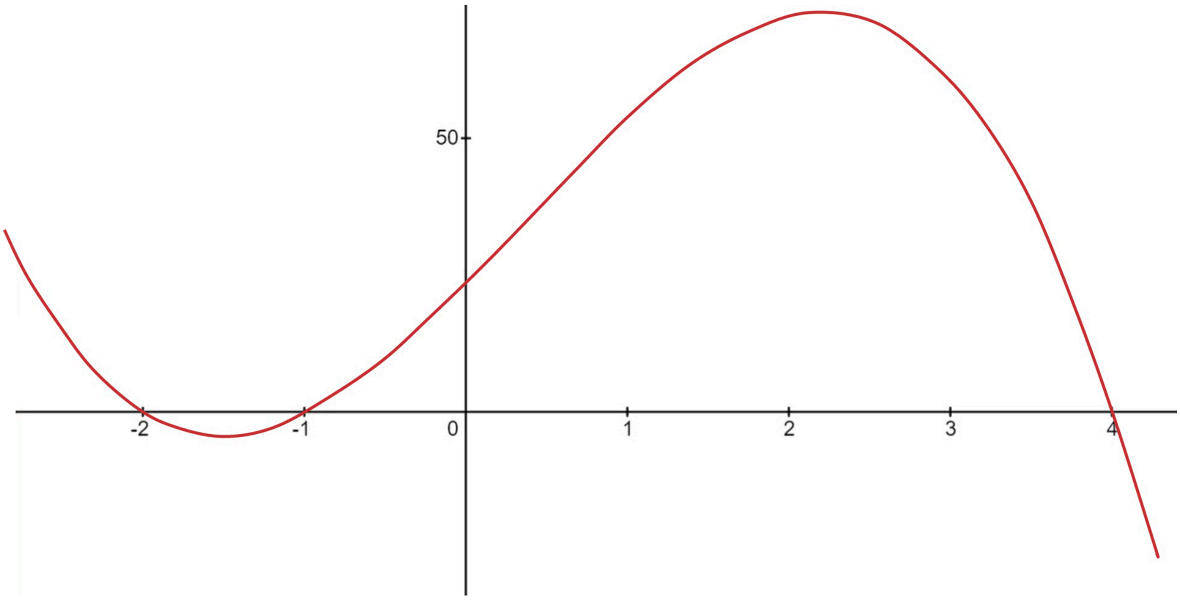
We can see from the graph that it has a y-intercept of 24, therefore, we need to multiply the whole function by 3 to give the positive cubic graph.

for multiplying by 3 [1]



The graph of $y = |p(x)|$ means means anything below the x-axis gets reflected above the x-axis.

We could also multiply the function by −3 to give the negative cubic graph.



As before, the graph of $y = |p(x)|$ means anything below the x-axis gets reflected above the x-axis.

Therefore, the final expression for the function would be

$$p(x) = \pm 3(x+2)(x+1)(x-4)$$ [1]

(3 marks)

5 Solve the inequality $|3x+2| > 8+x$.

Answer

The two possible options for the modulus function is a positive and a negative.

Solving the positive bracket

$$3x + 2 > 8 + x$$

Collecting like terms

$$2x > 6$$

$$x > 3$$

Solving the negative bracket

$$-(3x + 2) > 8 + x$$

Multiplying out the bracket

$$-3x - 2 > 8 + x$$

[1]

Collecting like terms

$$-10 > 4x$$

$$x < -\frac{5}{2}$$ [1]

$$x > 3$$ [1]

(3 marks)