IGCSE · Cambridge (CIE) · Further Maths

58 mins · 8 questions

Exam Questions

Factors of Polynomials

Operations with Polynomials / Polynomial Division / Factor & Remainder Theorem / Solving Cubic Equations

Medium (1 question)	/6
Hard (4 questions)	/30
Very Hard (3 questions)	/22
Total Marks	**/58**

Medium Questions

1 (a) $p(x) = 15x^3 + 22x^2 - 15x + 2$

Find the remainder when $p(x)$ is divided by $x + 1$.

Answer

Using the remainder theorem to find the value of $p(-1)$

$$p(-1) = 15(-1)^3 + 22(-1)^2 - 15(-1) + 2$$

[1]

Simplifying gives

$$p(-1) = -15 + 22 + 15 + 2$$

24 [1]

(2 marks)

(b) (i) Show that $x + 2$ is a factor of $p(x)$.

[1]

(ii) Write $p(x)$ as a product of linear factors.

[3]

Answer

i) Using the factor theorem, if $x - a$ is a factor of $f(x)$, then $f(a) = 0$

$$p(-2) = 15(-2)^3 + 22(-2)^2 - 15(-2) + 2$$

Simplifying each term

$$p(-2) = -120 + 88 + 30 + 2$$

$\mathbf{p(-2) = 0}$ [1]

ii) If $x + 2$ is a factor then factorising by inspection or using polynomial division gives

$$\begin{array}{r|rrrrl}
 & 15x^2 & -8x & +1 & & \\
x+2 & 15x^3 & +22x^2 & -15x & +2 & \\
 & 15x^3 & +30x^2 & -15x & +2 & - \\
\hline
 & & -8x^2 & -15x & & \\
 & & -8x^2 & -16x & & - \\
\hline
 & & & x & +2 & \\
 & & & x & +2 & - \\
\hline
 & & & & 0 &
\end{array}$$

$$(x + 2)(15x^2 - 8x + 1)$$

correct method to find the quadratic factor [1]

fully correct quadratic factor [1]

Factorising the second bracket

We need a pair of numbers that for $ax^2 + bx + c$, multiply to ac and add to b. In this case this is -3 and -5.

Rewrite the middle term using $-3x$ and $-5x$

$$15x^2 - 3x - 5x + 1$$

Group and factorise the first two terms, using $3x$ as the highest common factor, and group and factorise the second two terms, using -1 as the factor

$$3x(5x - 1) - 1(5x - 1)$$

Note that these terms now have a common factor of $(5x - 1)$ so this whole bracket can be factorised out, leaving $3x - 1$ in its own bracket.

This means that the quadratic factorises to

$$(3x - 1)(5x - 1)$$

Therefore, the cubic factorises to

$(x + 2)(3x - 1)(5x - 1)$ [1]

(4 marks)

Hard Questions

1 (a) The polynomial $\mathrm{p}(x) = 6x^3 + ax^2 + bx + 2$, where a and b are integers, has a factor of $x - 2$.
Given that $\mathrm{p}(1) = -2\mathrm{p}(0)$, find the values of a and b.

Answer

In this question you need to use the information given to form and then solve two linear simultaneous equations

$\mathrm{p}(x)$ has a factor $(x - 2)$, so from the factor theorem we know $x = 2$ is a solution of $\mathrm{p}(x) = 0$. Substitute $x = 2$ into $\mathrm{p}(x) = 0$

$$6(2)^3 + a(2)^2 + b(2) + 2 = 0$$

Now simplify to get an equation in terms of a and b.

$$6 \times 8 + 4a + 2b + 2 = 0$$
$$48 + 4a + 2b + 2 = 0$$

[1]

We can also collect like terms and pass the constant term to the right hand side

$$4a + 2b = -50$$

As all the terms are even, we can also divide both sides by 2

$$2a + b = -25 \quad \text{①}$$

We now have one of two simultaneous equations

To get the other, use "p(1) = −2p(0)". Substitute 1 into p to get p(1), substitute 0 into p to get p(0)

$$p(1) = 6(1)^3 + a(1)^2 + b(1) + 2$$
$$= 6 + a + b + 2$$
$$= 8 + a + b$$

$$p(0) = 6(0)^3 + a(0)^2 + b(0) + 2$$
$$= 2$$

p(1) = −2p(0) so equate p(1) to negative two times p(0)

$$8 + a + b = -2 \times 2$$

[1]

Simplify this and pass the constants to the right hand side to get the second of the two simultaneous equations

$$8 + a + b = -4$$
$$a + b = -12 \quad \textcircled{2}$$

Solve the simultaneous equations by elimination as below or by substitution; whichever method you feel most comfortable with!

$$2a + b = -25 \quad \textcircled{1}$$
$$a + b = -12 \quad \textcircled{2}$$

Subtract ② from ①, taking care with the negative numbers

$$\textcircled{1} - \textcircled{2}: \quad a = -13$$

any valid attempt to solve simultaneous equations [1]

And substitute $a = -13$ into ① or ② (② is easier) to obtain a

$$a + b = -12$$
$$-13 + b = -12$$
$$b = 1$$

Make sure that both the values of a and b are clearly written.

$a = -13,\ b = 1$ [1]

(4 marks)

(b) Using your values of a and b,

(i) find the remainder when $p(x)$ is divided by $2x - 1$

(ii) factorise $p(x)$.

Answer

i) From (a) we know that

$$p(x) = 6x^3 - 13x^2 + x + 2$$

The remainder theorem tells us that when p(x) is divided by (2*x* – 1), then the remainder is equal to $p\left(\frac{1}{2}\right)$.

So substitute $x = \frac{1}{2}$ into p:

$$p\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 - 13\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 2$$

[1]

Evaluate to find the remainder:

$$= 6\left(\frac{1}{8}\right) - 13\left(\frac{1}{4}\right)^2 + \frac{1}{2} + 2$$

$$= \frac{6}{8} - \frac{13}{4} + \frac{1}{2} + 2$$

$$= \frac{6}{8} - \frac{26}{8} + \frac{4}{8} + \frac{16}{8}$$

$$= \frac{0}{8}$$

0 [1]

ii) From (b)(i) we know that the remainder of $p\left(\frac{1}{2}\right)$ is 0.

Therefore:

$$(2x-1) \text{ is a factor}$$

[1]

We also know from the (a) that $(x-2)$ is a factor, therefore:

$$(2x-1)(x-2)(\square x+\square)=6x^3-13x^2+x+2$$

Looking at the constant terms in the factors (the numbers on their own inside the brackets), −1 times −2 equals 2, therefore the missing constant must be '1' And looking at the coefficients of x in the factors (the numbers next to x inside the brackets), 2 times 1 equals 2 so to multiply to '6' the missing coefficient of x must be '3'

$\mathbf{(2x-1)(x-2)(3x+1)}$ [1]

You can also use polynomial division

(4 marks)

2 (a) $p(x) = ax^3 + 3x^2 + bx - 12$ has a factor of $2x + 1$. When $p(x)$ is divided by $x - 3$ the remainder is 105.

Find the value of a and of b.

Answer

By the factor theorem, if $2x + 1$ is a factor, then $p\left(-\frac{1}{2}\right) = 0$ [this is obtained by solving $2x + 1 = 0$ to get $x = -\frac{1}{2}$]

Substitute this into the polynomial

$$p\left(-\frac{1}{2}\right) = a\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 12 = 0$$

$$-\frac{1}{8}a + \frac{3}{4} - \frac{1}{2}b - 12 = 0$$

Multiply through by 8

$$-a + 6 - 4b - 96 = 0$$

Collect the constant term on one side

$$a + 4b = -90$$

[1]

$p(3) = 105$ so use this to make a second equation

$$p(3) = a(3)^3 + 3(3)^2 + b(3) - 12 = 105$$

[1]

$$27a + 27 + 3b - 12 = 105$$

$$27a + 3b = 90$$

$$9a + b = 30$$

[1]

Now we have two equations which we can solve simultaneously

$$a + 4b = -90$$

$$9a + b = 30$$

Multiply the second equation by 4 and then subtract the equations

$$a + 4b = -90$$

$$36a + 4b = 120$$

Subtracting gives

$$35a = 210$$

$$a = 6$$

Find b

$$b = 30 - (9 \times 6) = -24$$

$a = 6,\ b = -24$ [2]

(5 marks)

(b) Using your values of a and b, write $\mathrm{p}(x)$ as a product of $2x + 1$ and a quadratic factor.

Answer

$a = 6,\ b = -24$ so $\mathrm{p}(x) = 6x^3 + 3x^2 - 24x - 12$

Rearrange $\mathrm{p}(x)$ so that the first 2 terms have a common factor of $2x$ and the second 2 terms have a common factor of 1

$$\mathrm{p}(x) = 6x^3 - 24x + 3x^2 - 12$$

A shortcut is to spot that the first two terms are equal to the last two terms multiply by $2x$. Take out the common factor of the first 2 terms and the second 2 terms

$$\mathrm{p}(x) = 2x(3x^2 - 12) + 1(3x^2 - 12)$$

Factorise

$$p(x) = (2x + 1)(3x^2 - 12)$$

$(2x + 1)(3x^2 - 12)$ [2]

1 mark for each term $3x^2$ *and* -12

(2 marks)

(c) Hence solve $p(x) = 0$.

Answer

$p(x) = (2x + 1)(3x^2 - 12)$ so we are solving $p(x) = 0$

$$(2x + 1)(3x^2 - 12) = 0$$

$$\text{either } (2x + 1) = 0 \text{ or } (3x^2 - 12) = 0$$

$$\text{if } (2x + 1) = 0 \text{ then } x = -\frac{1}{2}$$

$$\text{if } 3x^2 - 12 = 0 \text{ then } 3x^2 = 12$$

[1]

$$x^2 = 4$$

$$x = \pm 2$$

$x = -\frac{1}{2}, \ -2 \text{ or } 2$ [1]

(2 marks)

3 (a) $\mathrm{p}(x) = 6x^3 + ax^2 + 12x + b$, where a and b are integers.

$\mathrm{p}(x)$ has a remainder of 11 when divided by $x-3$ and a remainder of -21 when divided by $x+1$.
Given that $\mathrm{p}(x) = (x-2)Q(x)$, find $Q(x)$, a quadratic factor with numerical coefficients.

Answer

Using remainder theorem, $\mathrm{p}(3) = 11$ and $\mathrm{p}(-1) = -21$.

$$6(3)^3 + a(3)^2 + 12(3) + b = 11$$

$$6(-1)^3 + a(-1)^2 + 12(-1) + b = -21$$

[1]

Rearrange and simplify to get two equations in terms of a and b.

$$9a + b + 187 = 0$$

$$a + b + 3 = 0$$

[1]

Solve the simultaneous equations.

For example, by multiplying the second equation by 9 and using the elimination method.

$$\begin{array}{r} 9a + 9b + 27 = 0 \\ -\quad 9a + b + 187 = 0 \\ \hline 8b - 160 = 0 \end{array}$$

$$8b = 160$$
$$b = 20$$

[1]

Substitute $b = 20$ into one of the equations to find a.

$$a + 20 + 3 = 0$$
$$a = -23$$

[1]

Substitute these values into the original equation.

$$\mathrm{p}(x) = 6x^3 - 23x^2 + 12x + 20$$

Use algebraic (polynomial) long division to divide $\mathrm{p}(x)$ by $(x - 2)$.

$$\begin{array}{r|l} & 6x^2 - 11x - 10 \\ \hline x - 2 & 6x^3 - 23x^2 + 12x + 20 \\ & 6x^3 - 12x^2 \quad - \\ \hline & -11x^2 + 12x \\ & -11x^2 + 22x \quad - \\ \hline & -10x + 20 \\ & -10x + 20 \quad - \\ \hline & 0 \end{array}$$

using algebraic long division [1]

Therefore,

$$\mathrm{p}(x) = (x - 2)(6x^2 - 11x - 10)$$

$Q(x) = 6x^2 - 11x - 10$ [1]

(6 marks)

(b) Hence solve $\mathrm{p}(x) = 0$.

Answer

Factorise $6x^2 - 11x - 10$ in order to write $\mathrm{p}(x)$ as the product of three linear factors.

$$6x^2 - 11x - 10 = (3x + 2)(2x - 5)$$

[1]

Therefore,

$$p(x) = (x-2)(3x+2)(2x-5)$$

When $p(x) = 0$

$$(x-2)(3x+2)(2x-5) = 0$$

Solve.

$x = 2, \quad x = -\frac{2}{3}, \quad x = \frac{5}{2}$ [1]

(2 marks)

4 The three roots of $p(x) = 0$, where $p(x) = 2x^3 + ax^2 + bx + c$ are $x = \frac{1}{2}$, $x = n$ and $x = -n$, where a, b, c and n are integers. The y-intercept of the graph of $y = p(x)$ is 4. Find $p(x)$, simplifying your coefficients.

Answer

The y-intercept is 4 and this occurs when $x = 0$

Substitute $x = 0$ into $p(x)$

$$p(x) = 2(0)^3 + a(0)^2 + b(0) + c$$

$$p(x) = c = 4$$

Rewrite the polynomial

$$(1) \qquad p(x) = 2x^3 + ax^2 + bx + 4$$

Use the given roots to factorise the polynomial

$$p(x) = (x+n)(x-n)(2x-1)$$

[1]

Expand the first 2 brackets

$$p(x) = (x^2 - xn + xn - n^2)(2x-1)$$

Simplify

$$p(x) = (x^2 - n^2)(2x - 1)$$

[1]

Expand the brackets

$$(2) \qquad p(x) = 2x^3 - x^2 - 2n^2x + n^2$$

[1]

(1) and (2) are the same cubic so:

Compare constants:

$$4 = n^2$$

[1]

$$n = \pm 2$$

Compare coefficients of x^2

$$a = -1$$

Compare coefficients of x

$$b = -2n^2 = -2 \times 2^2 = -8$$

Write down $p(x)$

$p(x) = 2x^3 - x^2 - 8x + 4$ [1]

(5 marks)

Very Hard Questions

1 (a) The polynomial $\mathrm{p}(x) = ax^3 - 9x^2 + bx - 6$, where a and b are constants, has a factor of $x - 2$. The polynomial has a remainder of 66 when divided by $x - 3$. Find the value of a and of b.

Answer

Using the Factor Theorem, $\mathrm{p}(2) = 0$.

Therefore

$$0 = a(2)^3 - 9(2)^2 + b(2) - 6$$

[1]

Rearrange.

$$8a + 2b = 42$$

Using the Remainder Theorem, $\mathrm{p}(3) = 66$.

$$66 = a(3)^3 - 9(2)^2 + b(3) - 6$$

[1]

Rearrange.

$$27a + 3b = 153$$

Solve the simultaneous equations to find the values of a and b.

$$\begin{array}{rcrl} 54a & + & 6b & = 306 \\ -24a & + & 6b & = 126 \\ \hline 30a & & & = 180 \end{array}$$

$$a = 6$$

Substitute $a = 6$ into one of the equations to find the value of b.

$$8(6) + 2b = 42$$
$$b = -3$$

correct method to solve for both a and b [1]

$a = 6$ and $b = -3$ [1]

(4 marks)

(b) Using your values of a and b, show that $\mathrm{p}(x) = (x-2)\mathrm{q}(x)$, where $\mathrm{q}(x)$ is a quadratic factor to be found.

Answer

To work out the quadratic factor $q(x)$, we can use algebraic long division to divide $p(x)$ by $x-2$

$$\begin{array}{r|l} & 6x^2 \quad +3x \quad +3 \\ \hline x-2 & 6x^3 \quad -9x^2 \quad -3x \quad -6 \\ & 6x^3 - 12x^2 \qquad\qquad\qquad - \\ \hline & 3x^2 \quad -3x \\ & 3x^2 \quad -6x \qquad - \\ \hline & 3x \quad -6 \\ & 3x \quad -6 \quad - \\ \hline & 0 \end{array}$$

first two terms correct: $6x^2 + 3x...$[1]

$\mathrm{p}(x) = (x-2)(6x^2 + 3x + 3)$ [1]

(2 marks)

(c) Hence show that the equation $\mathrm{p}(x) = 0$ has only one real solution.

Answer

Using $\mathrm{q}(x)$ from part b, examine the discriminant to find any real roots.

$$b^2 - 4ac = (3)^2 - (4)(6)(3)$$
$$= -63$$
$$-63 < 0$$

[1]

In the quadratic formula, the discriminant is square rooted. Since we cannot take the square root of a negative number, a negative discriminant means no real roots.

Therefore, the only real solution to $p(x)$ is when $(x-2)=0$

$x=2$ is the only real root

$q(x) = 0$ has no real solutions, hence $p(x)$ only has one real solution [1]

(2 marks)

2 (a) $$p(x) = 2x^3 - 3x^2 - 23x + 12$$

Find the value of $p\left(\frac{1}{2}\right)$.

Answer

Substituting $x = \frac{1}{2}$ into the equation gives

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 23\left(\frac{1}{2}\right) + 12$$

$p\left(\frac{1}{2}\right) = 0$ [1]

(1 mark)

(b) Write $p(x)$ as the product of three linear factors and hence solve $p(x) = 0$.

Answer

Using the factor theorem, if $p\left(\frac{1}{2}\right) = 0$, then $(2x - 1)$ is a factor of $p(x)$.

To find the other factors, we need to use polynomial division to find the quadratic factor.

$$\begin{array}{r|l} & x^2 - x - 12 \\ 2x-1 & 2x^3 - 3x^2 - 23x + 12 \\ & 2x^3 - x^2 \qquad - \\ \hline & -2x^2 - 23x + 12 \\ & -2x^2 + x \qquad - \\ \hline & -24x + 12 \\ & -24x + 12 \qquad - \\ \hline & 0 \end{array}$$

Two correct terms in the quadratic factor [1]

Fully correct quadratic factor [1]

This means that

$$p(x) = (2x - 1)(x^2 - x - 12)$$

We need to factorise the quadratic bracket by finding a factor pair of −12 that has a sum of −1. This would be −4 and +3, therefore

$$p(x) = (2x - 1)(x + 3)(x - 4)$$

[1]

To solve, $p(x) = 0$ we find the value of x that would make the bracket equal to 0. In this case

$$x = \frac{1}{2}, -3, 4$$

Two correct solutions [1]
All three correct solutions [1]
(5 marks)

3 (a) The polynomial $p(x) = ax^3 + bx^2 - 19x + 4$, where a and b are constants, has a factor $x + 4$ and is such that $2p(1) = 5p(0)$.

Show that $p(x) = (x + 4)(Ax^2 + Bx + C)$, where A, B and C are integers to be found.

Answer

Find $p(1)$.

$$p(1) = a(1)^3 + b(1)^2 - 19(1) + 4$$
$$p(1) = a + b - 15$$

Therefore,

$$2p(1) = 2a + 2b - 30$$

Find $p(0)$.

$$p(0) = 4$$

Therefore,

$$5p(0) = 20$$

We are told that $2p(1) = 5p(0)$.

Therefore,

$$2a + 2b - 30 = 20$$

$$a + b - 15 = 10$$

$$a + b - 25 = 0$$

[1]

We are told that $(x + 4)$ is a factor of $p(x)$. Using factor theorem, this means that $p(-4) = 0$.

$$0 = a(-4)^3 + b(-4)^2 - 19(-4) + 4$$
$$0 = -64a + 16b + 80$$

Divide through by 16.

$$-4a + b + 5 = 0$$

[1]

We now have two equations in terms of a and b. Solve simultaneously.

$$\begin{aligned} -4a + b + 5 &= a + b - 25 \\ -5a &= -30 \\ a &= 6 \end{aligned}$$

Substitute into one of the equations to find a value for b.

$$\begin{aligned} -4(6) + b + 5 &= 0 \\ b - 19 &= 0 \\ b &= 19 \end{aligned}$$

1 mark for each value [2]

Substituting these values into the polynomial,

$$\mathrm{p}(x) = 6x^3 + 19x^2 - 19x + 4$$

We know one of the factors of the polynomial is $(x + 4)$. We can find the missing quadratic factor by inspection.

$$6x^3 + 19x^2 - 19x + 4 = (x + 4)(Ax^2 + Bx + C)$$

A must be 6 to achieve the $6x^3$. C must be 1 to achieve the 4.

$$6x^3 + 19x^2 - 19x + 4 = (x+4)(6x^2 + Bx + 1)$$

[1]

When expanded, the x^2 terms are $(4 \times 6x^2)$ and $(x \times Bx)$. Sum and equate to $19x^2$.

$$(24 + B)x^2 = 19x^2$$

Therefore,

$$24 + B = 19$$
$$B = -5$$

$(x+4)(6x^2 - 5x + 1)$ [1]

(6 marks)

(b) Hence factorise $p(x)$.

Answer

Factorise $6x^2 - 5x + 1$.

For example, by using the AC method.

$$6x^2 - 3x - 2x + 1$$

$$3x(2x-1) - 1(2x-1)$$

$$(3x-1)(2x-1)$$

$(x+4)(3x-1)(2x-1)$ [1]

(1 mark)

(c) Find the remainder when $p'(x)$ is divided by x.

Answer

Differentiate $6x^3 + 19x^2 - 19x + 4$

$$p'(x) = 18x^2 + 38x - 19$$

When $p'(x)$ is divided by x, we substitute 0 into the equation.

$$p'(0) = 18(0)^2 + 38(0) - 19$$
$$p'(0) = -19$$

-19 [1]

(1 mark)