IGCSE · Cambridge (CIE) · Further Maths

1 hour 15 questions

Exam Questions

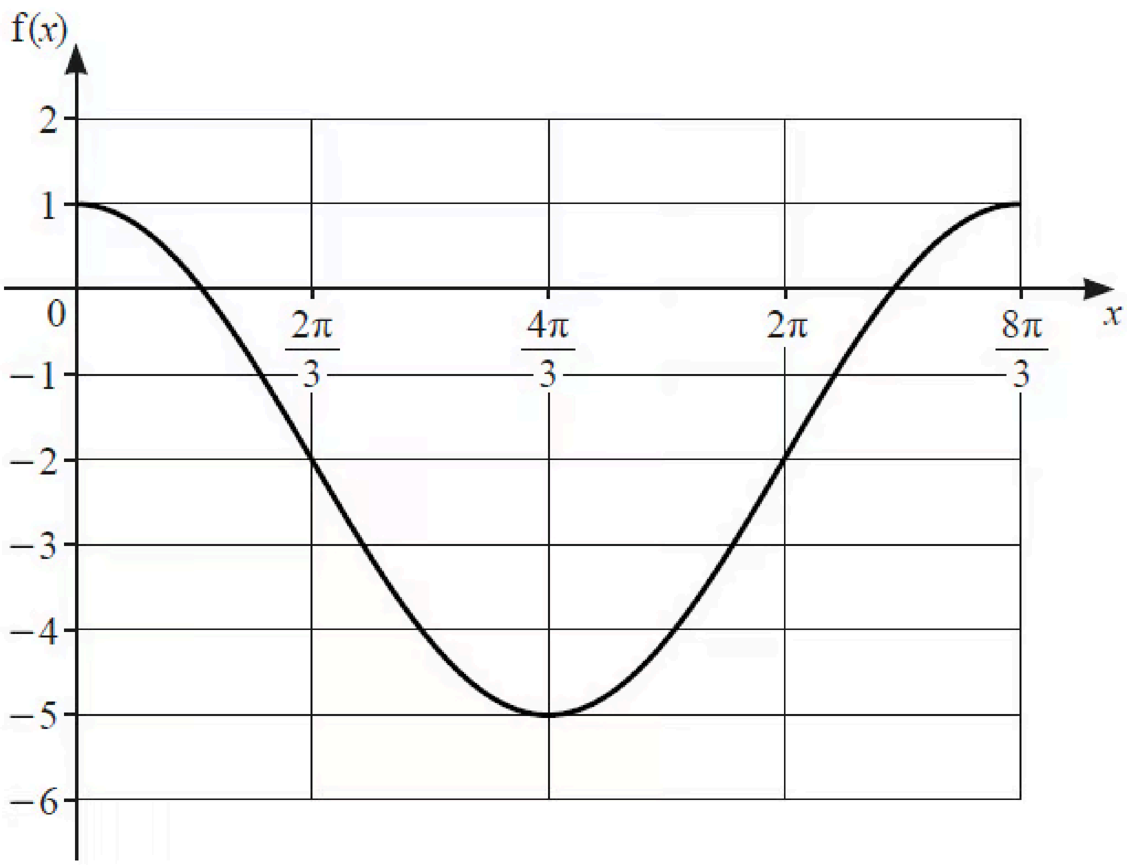
Functions

Language of Functions / Inverse Functions / Composite Functions

Medium (7 questions)	/34
Hard (5 questions)	/27
Very Hard (3 questions)	/26
Total Marks	**/87**

Medium Questions

1 (a)



The diagram shows the graph of $f(x) = a\cos bx + c$ for $0 \le x \le \dfrac{8\pi}{3}$ radians. Explain why f is a function.

Answer

Each value of x is mapped to a unique value of y [1]

(1 mark)

(b) Write down the range of f.

Answer

The range of f means the y values that the function can take

Look at the minimum and maximum points of the function

$-5 \leq f \leq 1$ [1]

(1 mark)

2 $g(x) = 3 + \frac{1}{x}$ for $x \geq 1$.

(i) Find an expression for $g^{-1}(x)$.

[2]

(ii) Write down the range of g^{-1}.

[1]

(iii) Find the domain of g^{-1}.

[2]

Answer

i) Let $y = 3 + \frac{1}{x}$

Rearrange the equation to make x the subject

Subtract 3

$$y - 3 = \frac{1}{x}$$

Make x the subject

$$x = \frac{1}{y-3}$$

[1]

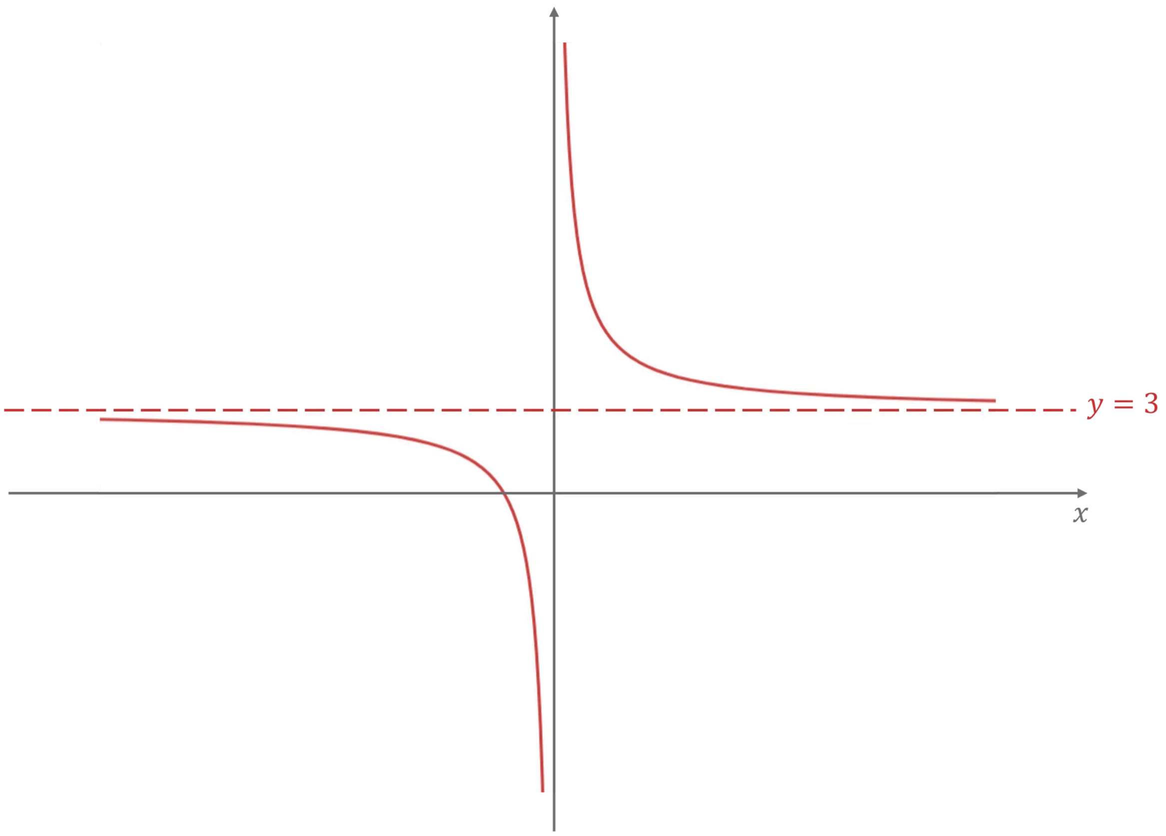
$g^{-1}(x) = \frac{1}{x-3}$ [1]

ii) The range of g^{-1} is the domain of g

$g^{-1} \geq 1$ [1]

iii) The domain of g^{-1} is equal to the range of g

To work out the range of g, consider what the graph without a restriction on the domain will look like



However, the domain is restricted to $x \geq 1$ so work out y at the point when $x = 1$

$$y = 3 + \frac{1}{1} = 4$$

When x is greater than this, the graph tends towards the asymptote so the minimum of the graph is when $y = 3$ but it cannot be equal to this point due to the asymptote

So the range of g is $3 < g \leq 4$ which means the domain of g^{-1} is

$3 < x \leq 4$

correct numbers [1]
correct inequalities [1]
(5 marks)

3 (a) It is given that $f(x) = 5\ln(2x+3)$ for $x > -\dfrac{3}{2}$.

Write down the range of f.

Answer

The domain is given by $x > -\dfrac{3}{2}$ and this would prevent taking a natural log of a negative number or 0. This means that all other numbers are valid and the output could be any real number.

$f \in \mathbb{R}$ [1]
(1 mark)

(b) Find f^{-1} and state its domain.

Answer

To find the inverse function, we set the function equal to y, rearrange to make x the subject, and then "swap" the variables.

Set the function equal to y

$$y = 5\ln(2x + 3)$$

Divide both sides by 5

$$\frac{y}{5} = \ln(2x + 3)$$

Take exponentials of both sides

$$e^{\frac{y}{5}} = 2x + 3$$

[1]

Make x the subject

$$\frac{e^{\frac{y}{5}} - 3}{2} = x$$

"Swap" the variables

$$\frac{e^{\frac{x}{5}} - 3}{2} = y$$

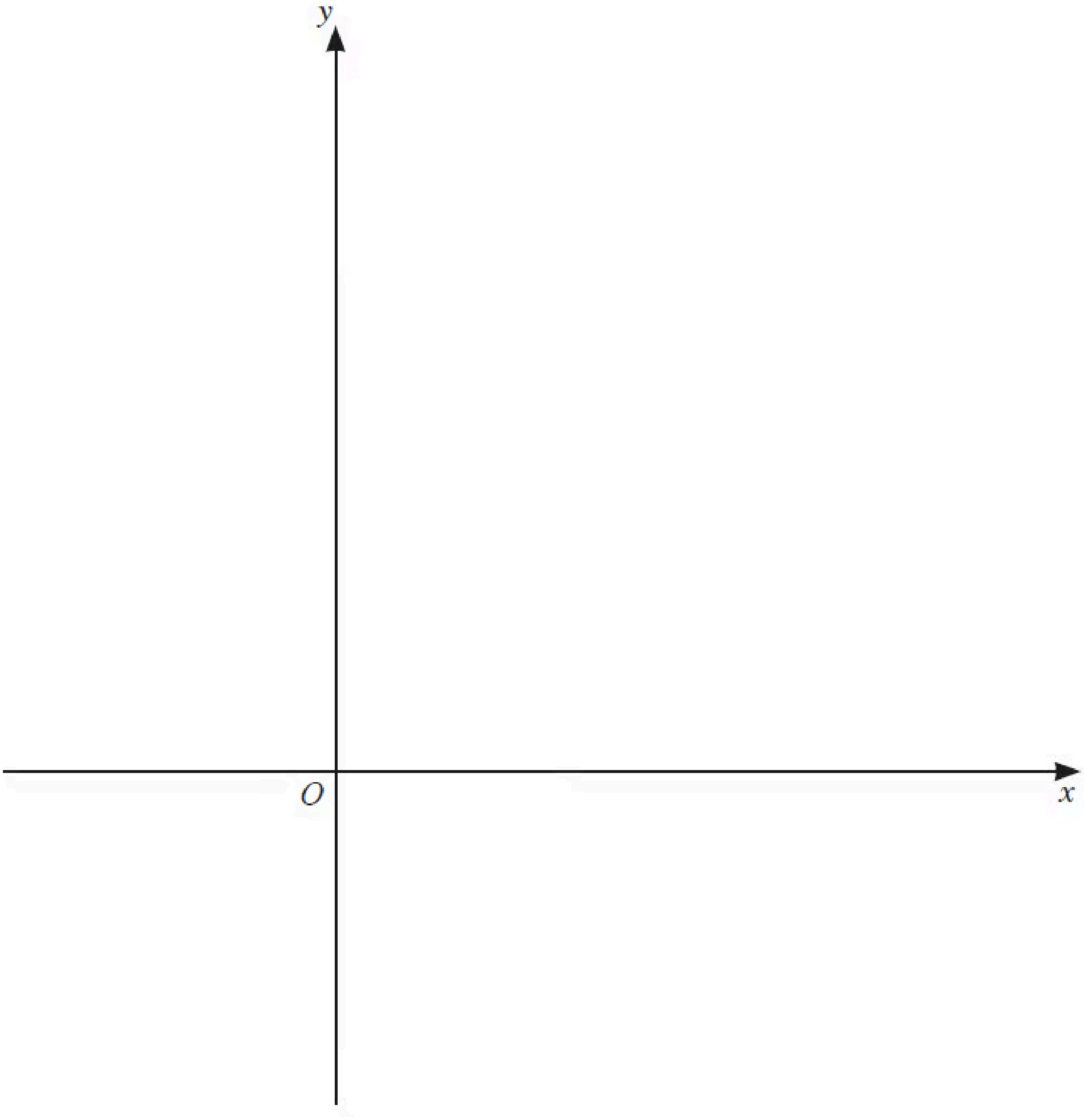
The domain of an inverse function is the range of the original function, therefore

$$f^{-1}(x) = \frac{e^{\frac{x}{5}} - 3}{2}$$ [1]

$x \in \mathbb{R}$ [1]

(3 marks)

(c) On the axes below, sketch the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$. Label each curve and state the intercepts on the coordinate axes.



Answer

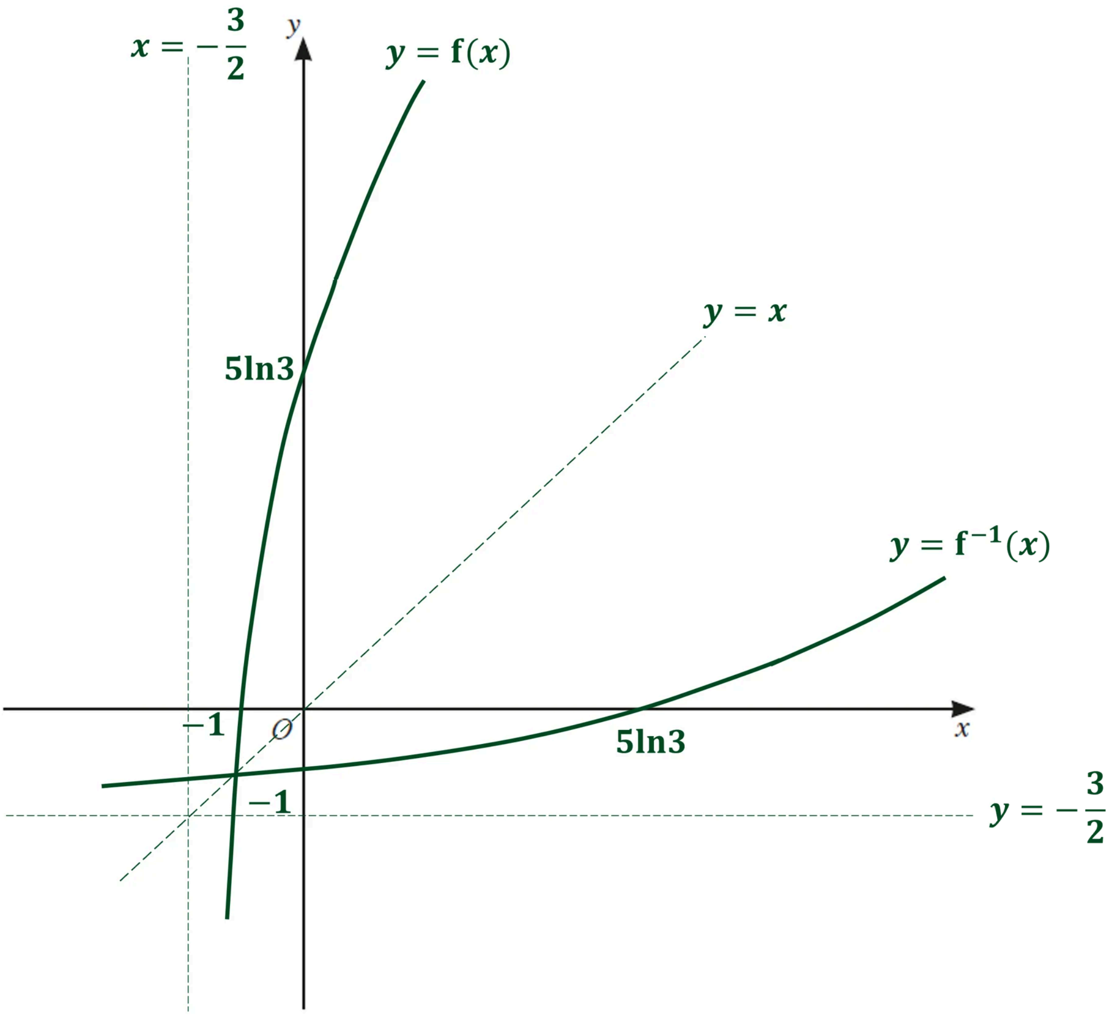
One of the key features of a log graph is that is has an x-intercept at (1 , 0) and an asymptote at $x = 0$. We need to apply a horizontal translation in the negative direction of 3 units, a horizontal stretch of scale factor $\frac{1}{2}$, and a vertical stretch of scale factor 5.

This means that the x-intercept is now (-1 , 0) and the asymptote is now $x = -\frac{3}{2}$.

The y-intercept occurs when $x = 0$, so in this case

$$y = 5\ln(0 + 3) = 5\ln3$$

The inverse function is a reflection in the line $y = x$.



correct shape of $y = \mathrm{f}(x)$ [1]

correct shape of $y = \mathrm{f}^{-1}(x)$ [1]

correct intercepts for $y = \mathrm{f}(x)$ [1]

correct intercepts for $y = \mathrm{f}^{-1}(x)$ [1]

correct line of symmetry [1]

(5 marks)

4 $$\mathrm{f}(x) = 4\ln(2x - 1)$$

(i) Write down the largest possible domain for the function f.

[1]

(ii) Find $f^{-1}(x)$ and its domain.

[3]

Answer

(i) Since you cannot take the logarithm of a negative number, $2x-1$ must be greater than 0.

Therefore, the domain of $f(x)$ is

$$x > \frac{1}{2}$$ [1]

(ii) To find the inverse of $y = 4\ln(2x-1)$, switch x and y, then rearrange to make y the subject.

$$x = 4\ln(2y-1)$$

$$\frac{x}{4} = \ln(2y-1)$$

Exponentiate both sides and rearrange.

$$e^{\frac{x}{4}} = 2y-1$$

$$e^{\frac{x}{4}} + 1 = 2y$$

$$y = \frac{1}{2}\left(e^{\frac{x}{4}} + 1\right)$$

[1]

$$f^{-1}(x) = \frac{1}{2}(e^{\frac{x}{4}} + 1) \; [1]$$

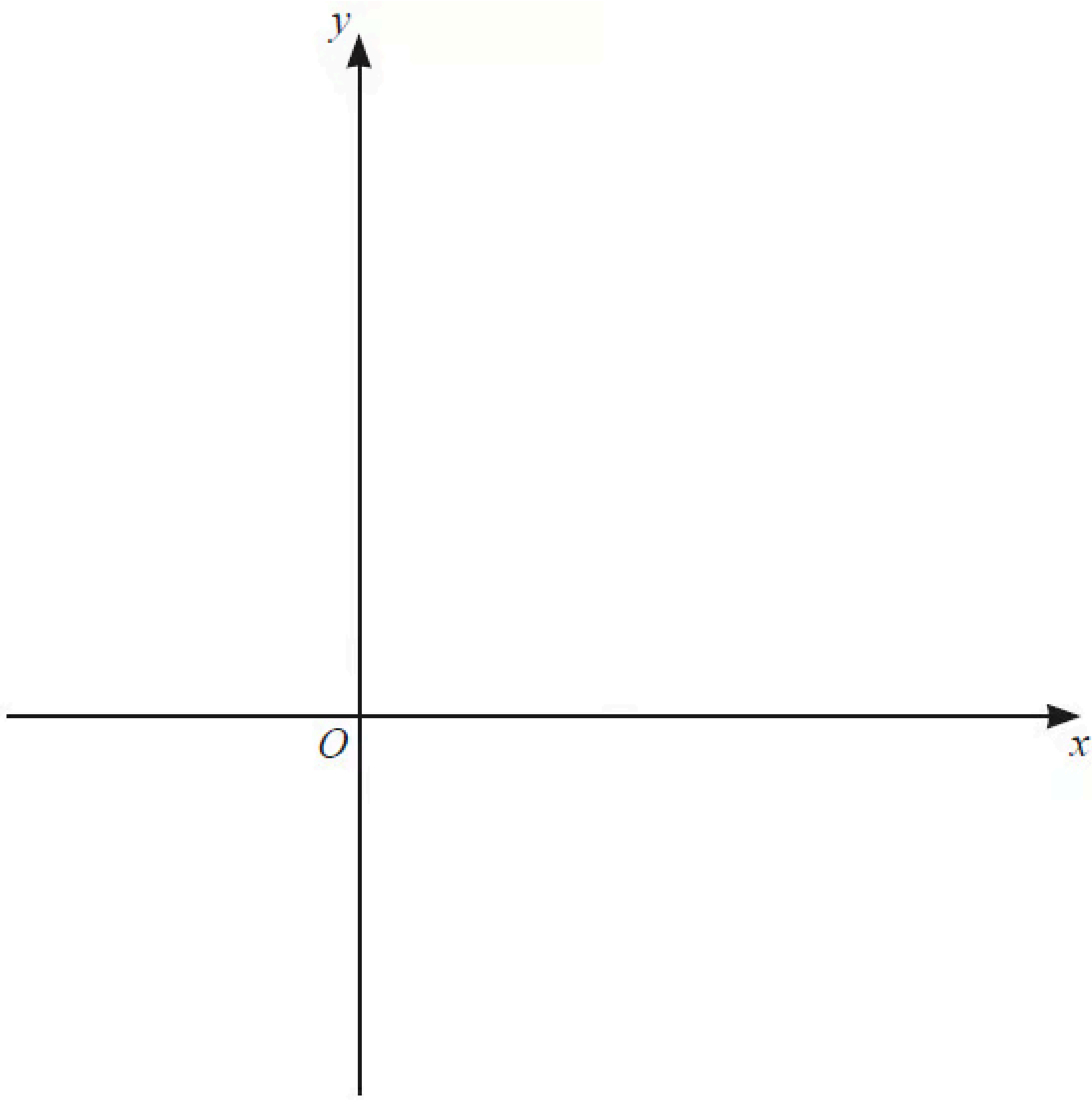
x can be any real number, therefore the domain is

$$x \in \mathbb{R} \; [1]$$

(4 marks)

5 $h(x) = 2\ln(3x - 1)$ for $x \geq \frac{2}{3}$.

The graph of $y = h(x)$ intersects the line $y = x$ at two distinct points. On the axes below, sketch the graph of $y = h(x)$ and hence sketch the graph of $y = h^{-1}(x)$.



Answer

Consider the graph of $\mathrm{h}(x) = 2\ln(3x - 1)$

It is a transformation of the graph $y = \ln(x)$

Find the x-intercept of $\mathrm{h}(x)$ i.e. when $\mathrm{h}(x) = 0$

$$2\ln(3x - 1) = 0$$

Solve for x

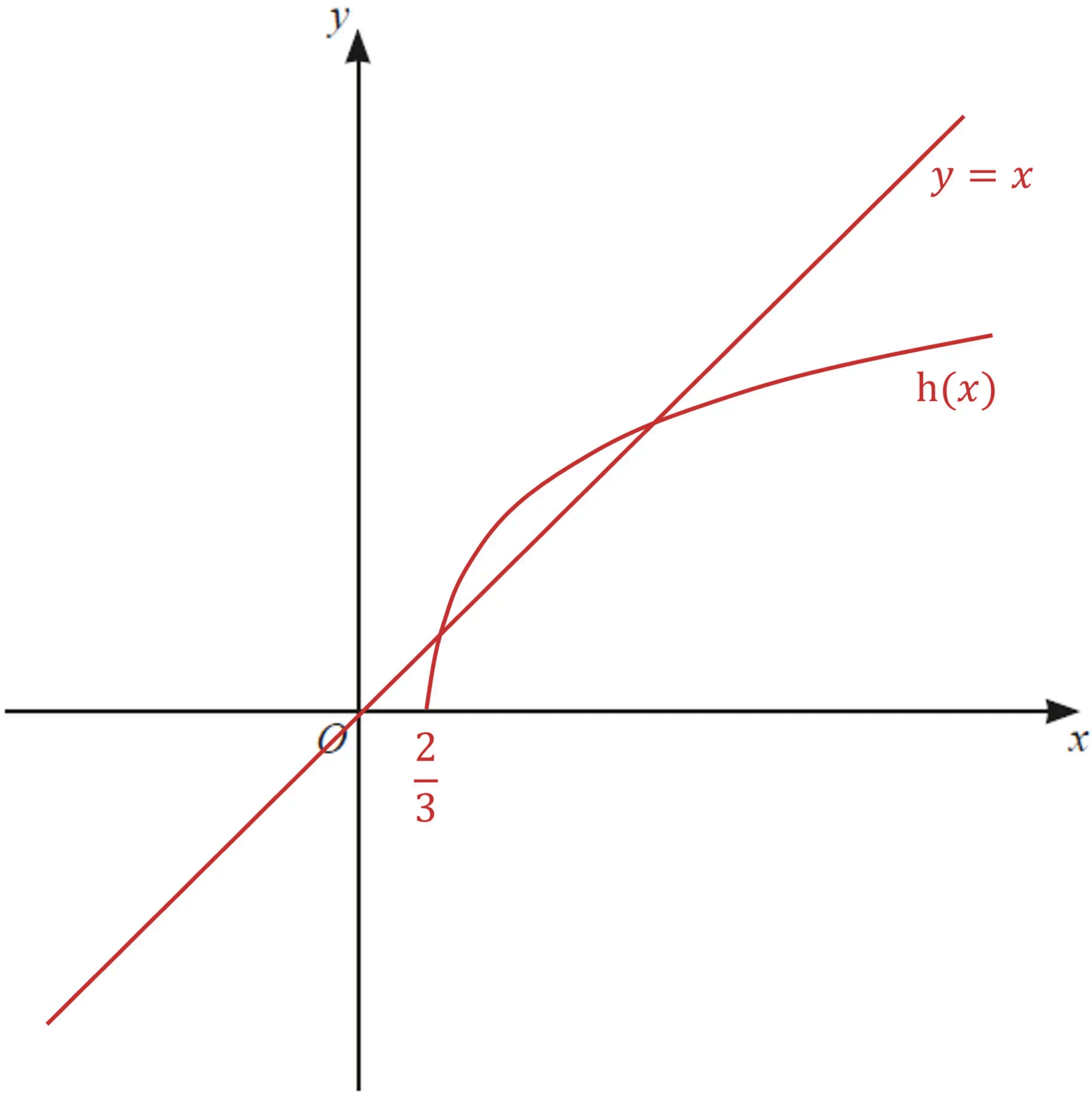
$$\ln(3x - 1) = 0$$

$$3x - 1 = 1$$

$$x = \frac{2}{3}$$

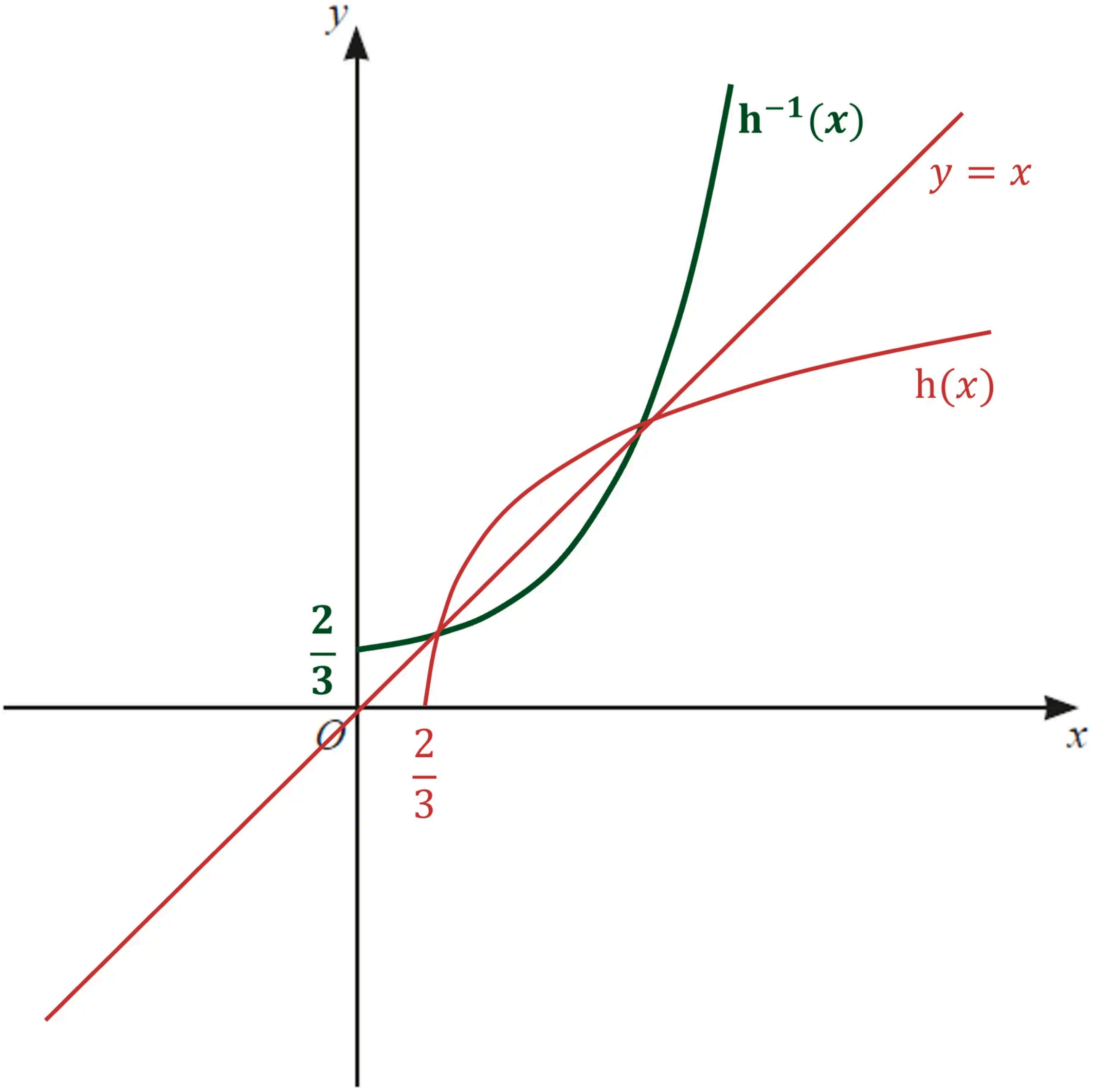
Since the domain of $\mathrm{h}(x)$ is $x \geq \frac{2}{3}$, the graph will start from this point

Draw a sketch of $\mathrm{h}(x)$ and $y = x$



[1]

Now draw h^{-1} which will be a reflection of h in the line $y = x$



reflection in the line $y = x$ [1]

both graphs drawn over correct domain [1]

correct graphs intersecting twice [1]

(4 marks)

6 (a) Two functions are given by $f(x) = 1 + 8x$ where $x \in \mathbb{R}$ and $g(x) = 2 + \ln x$ where $x > 0$.

Find an expression for $f^{-1}(x)$.

Answer

Method 1

It is possible to see the answer by undoing the function

subtract one then divide by 8

$$f^{-1}(x) = \frac{x-1}{8}$$

[B1]

Method 2

Set $y = \ldots$ to the function

$$y = 1 + 8x$$

swap x and y, then make y the subject

$$x = 1 + 8y$$
$$x - 1 = 8y$$
$$\frac{x-1}{8} = y$$

Write the answer as $f^{-1}(x) = \ldots$

$$f^{-1}(x) = \frac{x-1}{8}$$

[B1]

(1 mark)

(b) State the geometric relationship between the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$.

Answer

The graph of $y = f^{-1}(x)$ is a reflection in the line $y = x$ of the graph $y = f(x)$

[B1]

Mark Scheme and Guidance

Your explanation must state the equation of the line of reflection $(y = x)$ to get this mark.

(1 mark)

(c) Find $\mathrm{fg}^2(1)$.

Answer

This means $\mathrm{fgg}(1)$ which is $\mathrm{g}(1)$ first, then g of the answer, then f of that answer

$$\mathrm{g}(1) = 2 + \ln 1$$

Remember that $\log_a 1 = 0$

$$\mathrm{g}(1) = 2$$

Now do $\mathrm{g}(2)$

$$\mathrm{g}(2) = 2 + \ln 2$$

[M1]

Now do $\mathrm{f}(2 + \ln 2)$ and simplify

$$\begin{aligned} \mathrm{f}(2 + \ln 2) &= 1 + 8(2 + \ln 2) \\ &= 1 + 16 + 8\ln 2 \end{aligned}$$

$17 + 8\ln 2$

[A1]

Mark Scheme and Guidance

M1: For an attempt to substitute 1 into $\mathrm{g}(x)$ twice (you do not need to simplify working for this mark).

A1: For the correct final answer, fully simplified in terms of $\ln 2$.

(2 marks)

(d) Explain why the function $\mathrm{gf}(x)$ does not exist.

Answer

The function $\mathrm{f}(x)$ allows negative inputs, which can give negative outputs in the range

$$\text{e.g. } \mathrm{f}(-1) = 1 + 8(-1) = -7$$

These negative outputs from f become negative inputs in g

But $\mathrm{g}(x)$ only accepts positive inputs (as its domain is $x > 0$)

The range of $\mathrm{f}(x)$ does not lie within the domain of $\mathrm{g}(x)$

[B1]

Mark Scheme and Guidance

You must refer to both the 'range of $\mathrm{f}(x)$' and the 'domain of $\mathrm{g}(x)$' in your explanation to get this mark.

(1 mark)

7 (a) A curve has the equation $y = 5x^2 - 10x + 9$.

Write the equation in the form $y = p(x+q)^2 + r$ where p, q and r are constants to be found.

Answer

To complete the square, first factorise out a 5 from first two terms

$$y = 5[x^2 - 2x] + 9$$

Complete the square on $x^2 - 2x$ by halving the -2 and writing it as $(x-1)^2 - (-1)^2$

$$(x-1)^2 - 1$$

Substitute that back into the original expression

$$y = 5[(x-1)^2 - 1] + 9$$

Expand through by the 5 (without expanding the brackets)

$$y = 5(x-1)^2 - 5 + 9$$

Simplify the final constant

$y = 5(x-1)^2 + 4$

[B1 B1]

Mark Scheme and Guidance

B1: For at least $5(x-1)^2 + \ldots$ in your answer (may have the wrong constant).

B1: For $5(x-1)^2 + 4$ fully correct (does not need "$y =$ " for this mark).

(2 marks)

(b) Hence find the coordinates of the turning point on the curve.

Answer

The turning point on any curve in the form $y = p(x+q)^2 + r$ is always $(-q, r)$

For $y = 5(x-1)^2 + 4$, the turning point is at $(1, 4)$

$(1, 4)$

[B1]

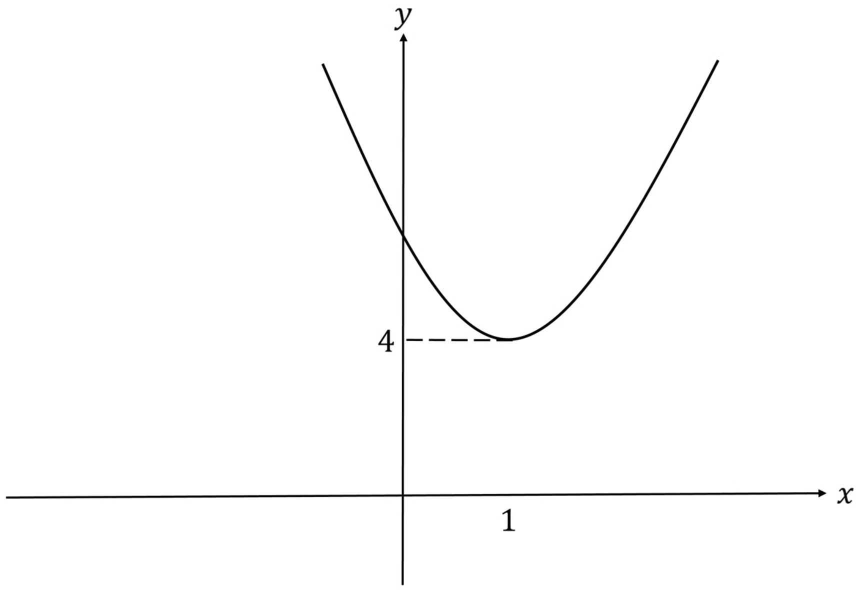
(1 mark)

(c) Find the range of the function $\mathrm{f}(x) = 5x^2 - 10x + 9$ where $x \in \mathbb{R}$.

Answer

The graph $y = 5x^2 - 10x + 9$ is a positive quadratic (U-shape)

This makes the turning point $(1, 4)$ from part (b) a minimum point



That means all outputs (y-coordinates) from $\mathrm{f}(x)$ are greater than or equal to 4

$\mathrm{f}(x) \geq 4$

[B1]

(1 mark)

(d) Explain why the inverse function $\mathrm{f}^{-1}(x)$ does not exist.

Answer

A function can only have an inverse if it is one–to–one

$f(x) = 5x^2 - 10x + 9$ is many–to–one

$f(x)$ **is not one-to-one, so it cannot have an inverse function**

[B1]

Mark Scheme and Guidance

Your answer must refer to the function not being 'one-to-one' to get this mark.

Writing '$f(x)$ is many-to-one' does not score the mark.

Writing '$f(x)$ is many-to-one but functions must be one-to-one for their inverses to exist' does score the mark.

(1 mark)

Hard Questions

1 It is given that $h(x) = a + \dfrac{b}{x^2}$, where a and b are non-zero constants.

(i) Explain why $-2 \leqslant x \leqslant 2$ is not a suitable domain for $h(x)$.

(ii) Given that $h(1) = 4$ and $h'(1) = 16$, find the values of a and b.

Answer

i) We can't divide by 0; so at $x = 0$, $h(x) = a + \dfrac{b}{0}$ which is undefined

the function is undefined at $x = 0$ [1]

ii) We will form simultaneous equations in a and b from the information given $h(1) = 4$ so for the first equation,

$$a + \frac{b}{1^2} = 4$$
$$a + b = 4$$

$h'(1) = 16$ so we need to differentiate h(*x*). Start by rewriting $\dfrac{b}{x^2}$ as a negative power of *x*

$$\frac{b}{x^2} = bx^{-2}$$

Now differentiate h(*x*). Remember that a and b are just constants like any others

$$h(x) = a + bx^{-2}$$
$$h'(x) = -2bx^{-3}$$

correct derivative as well as '$a + b = 4$' [1]

Now form a second equation from $h'(1) = 16$

$$-2b(1)^{-3} = 16$$

As a was eliminated by the differentiation, we can simplify and solve for b directly

$$-2b = 16$$
$$b = -8$$

Substitute $b = -8$ into $a + b = 4$ and solve for a

$$a - 8 = 4$$

a = 12 [1]

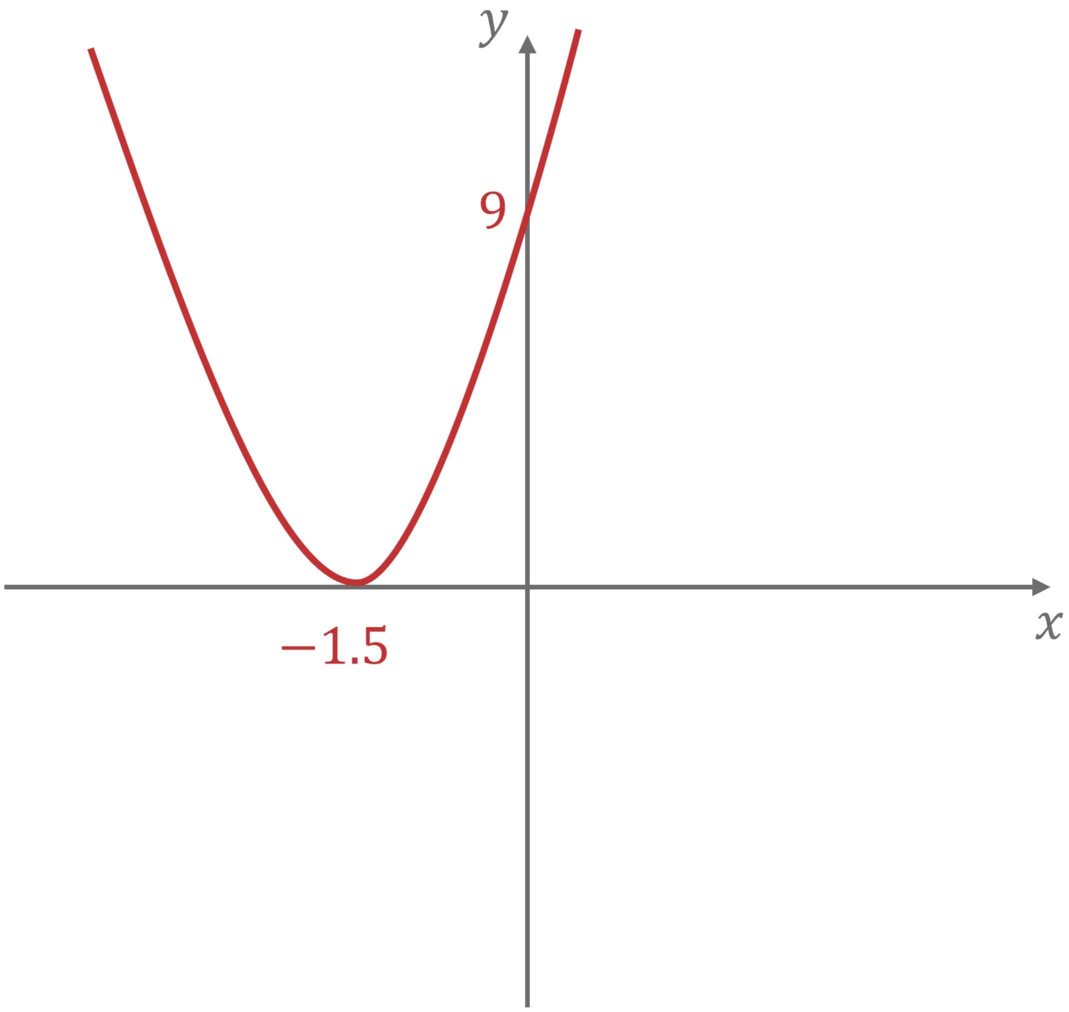
(3 marks)

2 (a) $\mathrm{f} : x \mapsto (2x+3)^2$ for $x > 0$

Find the range of f.

Answer

Consider the graph of the function



The domain is $x > 0$ so the part of the graph we want is the part to the right of the y–axis

When $x = 0$

$$(2x + 3)^2 = 3^2 = 9$$

We want the section of the graph to the right of this so the range is $\mathrm{f} > 9$

f > 9 [1]
(1 mark)

(b) Explain why f has an inverse.

Answer

Only one-one functions have an inverse

Since the domain has been restricted by $x > 0$, f is now one-one so it does have an inverse (without the restriction it would not have had an inverse because this would be a many-one function)

It is a one-one function because of the restricted domain [1]

(1 mark)

(c) Find f^{-1}.

Answer

Let $y = (2x+3)^2$

[1]

Rearrange to make x the subject

Square root both sides

$$\pm\sqrt{y} = 2x + 3$$

Subtract 3

$$\pm\sqrt{y} - 3 = 2x$$

Divide by 2

$$x = \frac{\pm\sqrt{y} - 3}{2}$$

[1]

Replace the x with f^{-1} and the y with x

Take the positive square root as range of the inverse is the same as the domain of f which includes only positive values

$$\mathbf{f^{-1} = \frac{\sqrt{x} - 3}{2}}$$ [1]

(3 marks)

(d) State the domain of f^{-1}.

Answer

The domain of f^{-1} is the same as the range of f (but remember that domains are written in terms of x)

$x > 9$ [1]

(1 mark)

(e) Given that $\mathrm{g} : x \mapsto \ln(x+4)$ for $x > 0$, find the exact solution of $\mathrm{fg}(x) = 49$.

Answer

Method 1

$\mathrm{f} : x \mapsto (2x+3)^2$ for $x > 0$

$\mathrm{g} : x \mapsto \ln(x+4)$ for $x > 0$

Find $\mathrm{fg}(x)$

$$\mathrm{fg}(x) = \mathrm{f}(\mathrm{g}(x))$$

$$\mathrm{fg}(x) = \mathrm{f}(\ln(x+4))$$

[1]

$$\mathrm{fg}(x) = (2\ln(x+4)+3)^2$$

Use that $\mathrm{fg}(x) = 49$

$$(2\ln(x+4)+3)^2 = 49$$

Square root

$$2\ln(x+4)+3 = 7$$

Subtract 3

$$2\ln(x+4) = 4$$

Divide by 2

$$\ln(x+4)=2$$

[1]

Take the exponential of both sides to undo the natural log

$$x+4=e^2$$

$$x=e^2-4$$

$\boldsymbol{x=e^2-4}$ [1]

Method 2

$f : x \mapsto (2x+3)^2$ for $x > 0$

$g : x \mapsto \ln(x+4)$ for $x > 0$

Rearrange $fg(x)=49$ by taking the inverse of f

$$g(x)=f^{-1}(49)$$

Substitute 49 into the expression for the inverse of f

$$g(x)=\frac{\sqrt{49}-3}{2}=2$$

[1]

use the expression for the function g

$$\ln(x+4)=2$$

[1]

Take the exponential of both sides to undo the natural log

$$x+4=e^2$$

$$x=e^2-4$$

$\boldsymbol{x=e^2-4}$ [1]

(3 marks)

3

$$g(x) = x + 5 \text{ for } x \in \mathbb{R}$$
$$h(x) = \sqrt{2x-3} \text{ for } x \geq \frac{3}{2}$$

Solve $gh(x) = 7$.

Answer

Substitute $h(x)$ into $g(x)$

$$gh(x) = \sqrt{2x-3} + 5$$

Equate to 7.

$$\sqrt{2x-3} + 5 = 7$$

[1]

Rearrange to make x the subject.

$$\sqrt{2x-3} = 2$$
$$2x - 3 = 2^2$$
$$2x = 4 + 3$$

[1]

$x = \frac{7}{2}$ [1]

(3 marks)

4 (a) The functions f and g are defined as follows.

$$f(x) = x^2 + 4x \text{ for } x \in R$$
$$g(x) = 1 + e^{2x} \text{ for } x \in R$$

Find the range of f.

Answer

Identify the $y$coordinate of turning point by completing the square for $f(x) = x^2 + 4x$.

$$(x+2)^2 - (2)^2$$

$$= (x+2)^2 - 4$$

$$(-2,\ -4)$$

[1]

Since the coefficient of x^2 is positive, this turning point is the **minimum** point.

$\mathbf{f(x) \geq -4}$ [1]

(2 marks)

(b) Write down the range of g.

Answer

For e^{2x}, as $x \to -\infty$, $e^{2x} \to 0$

e^{2x} must be greater than 0, but will never reach 0, and therefore the function must be greater than 1.

$\mathbf{g(x) > 1}$ [1]

(1 mark)

(c) Find the exact solution of the equation $fg(x) = 21$, giving your answer as a single logarithm

Answer

Substitute $g(x)$ into $f(x)$, expand and simplify to get $fg(x)$.

$$(1+e^{2x})^2 + (4)(1+e^{2x})$$

$$= (1+e^{2x})(1+e^{2x}) + 4 + 4e^{2x}$$
$$= 1 + 2e^{2x} + e^{4x} + 4 + 4e^{2x}$$
$$= e^{4x} + 6e^{2x} + 5$$

[1]

Factorise $\mathrm{fg}(x) = 21$.

$$e^{4x} + 6e^{2x} + 5 = 21$$

$$e^{4x} + 6e^{2x} - 16 = 0$$

$$(e^{2x} + 8)(e^{2x} - 2) = 0$$

[1]

The first bracket will lead us to $e^{2x} = -8$. Since we cannot take a logarithm of a negative number, we cannot use this bracket to achieve a solution. We can use the second bracket.

$$e^{2x} - 2 = 0$$

$$e^{2x} = 2$$

[1]

Take the natural log of both sides and solve.

$$\ln e^{2x} = \ln 2$$

$$2x(\ln e) = \ln 2$$

$$2x = \ln 2$$

$$x = \frac{1}{2}\ln 2$$

$x = \ln(2^{\frac{1}{2}})$ [1]

(4 marks)

5 (a) $$f(x) = x^2 + 2x - 3 \text{ for } x \geq -1$$

Given that the minimum value of $x^2 + 2x - 3$ occurs when $x = -1$, explain why $f(x)$ has an inverse.

Answer

We are told that the function has a domain of $x \geq -1$. So it is only only half of a quadratic graph.

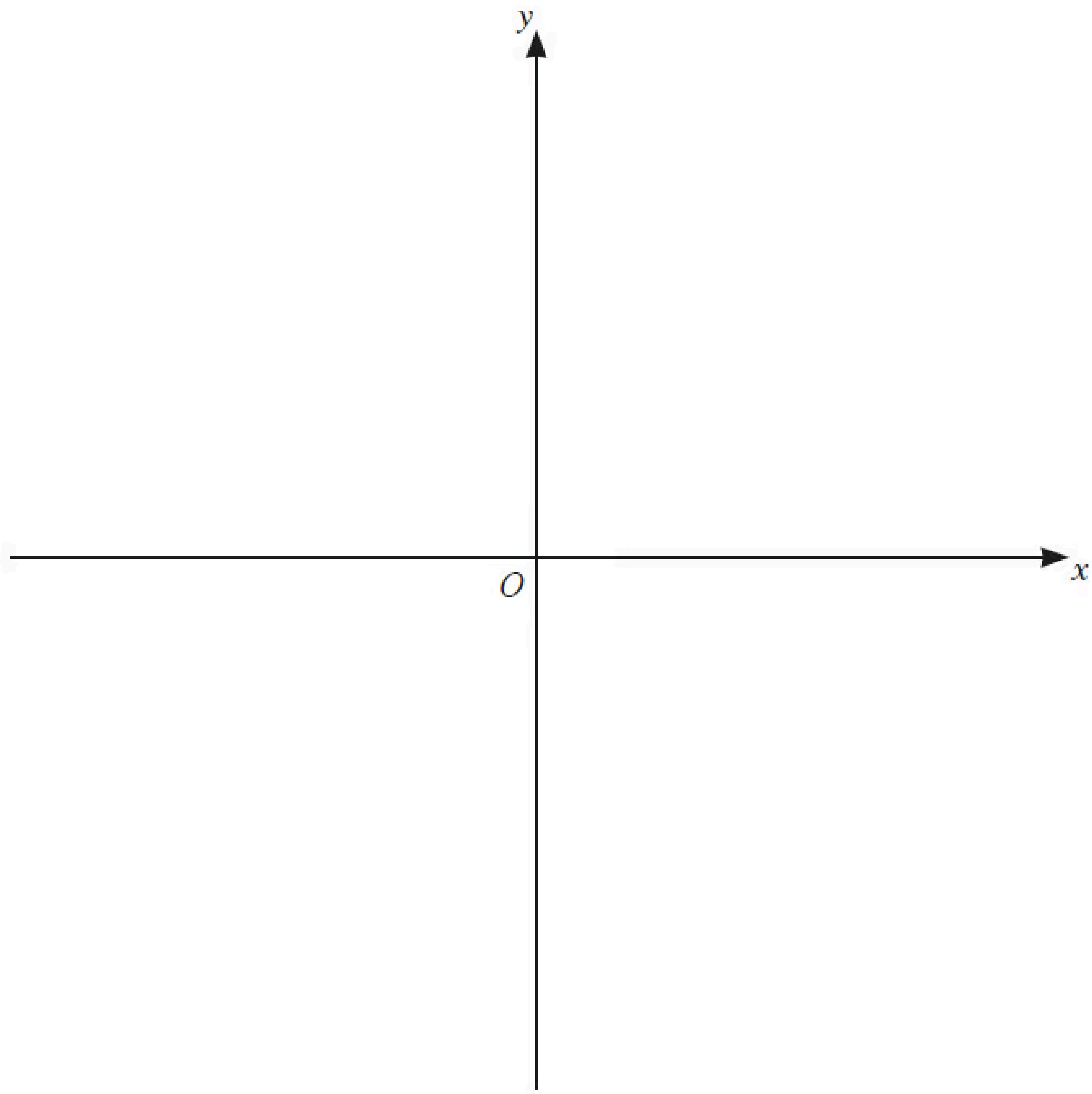
This means that we have a one-one function, since every point y will only have one value of x.

One-one functions have inverse functions.

It is a one-one function because the function is always increasing for the given restricted domain [1]

(1 mark)

(b) On the axes below, sketch the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$. Label each graph and state the intercepts on the coordinate axes.



Answer

We are told in part a that the minimum for $f(x)$ occurs when $x = -1$.

Substitute $x = -1$ into the equation to find the corresponding y value.

$$y = (-1)^2 + (2)(-1) - 3$$
$$y = -4$$

Substitute $x = 0$ into the equation to find the y intercept.

$$y = -3$$

Solve the equation to find the point where the curve crosses the x axis.

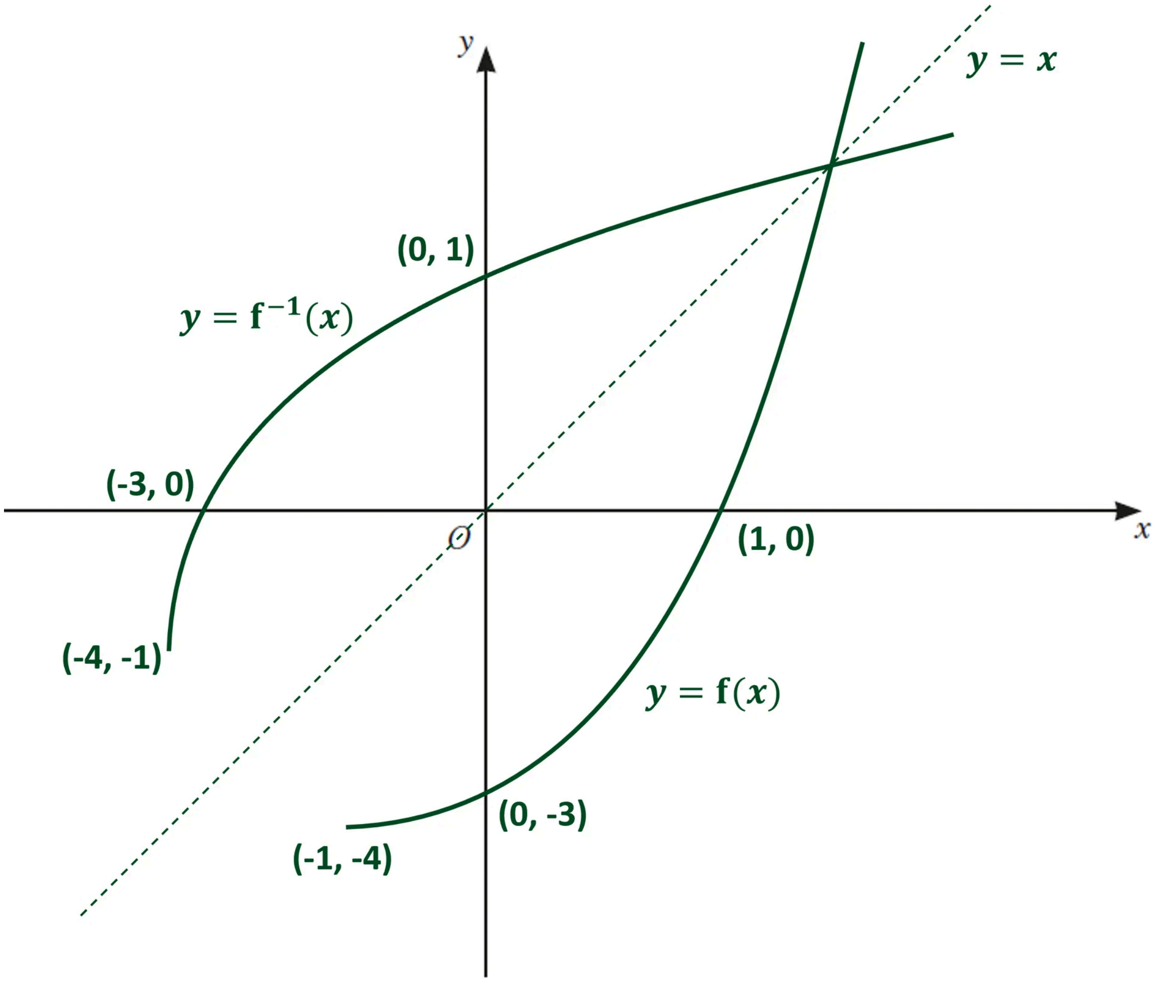
$$(x+3)(x-1) = 0$$

$$x = -3 \text{ and } x = 1$$

We can exclude $x = -3$ because of the given domain.

Therefore, we need to draw half a parabola starting a $(-1, -4)$ that passes through $(1, 0)$ and $(0, -3)$

The inverse function of $\mathrm{f}(x)$ can be drawn by reflecting this in the line $y = x$



correct graph of $y = \mathrm{f}(x)$ [1]

correctly labelling (0,-3) and (1,0) [1]

correct graph of $y = \mathrm{f}^{-1}(x)$ [1]

correctly labelling (-3,0) and (0,1) [1]

(4 marks)

Very Hard Questions

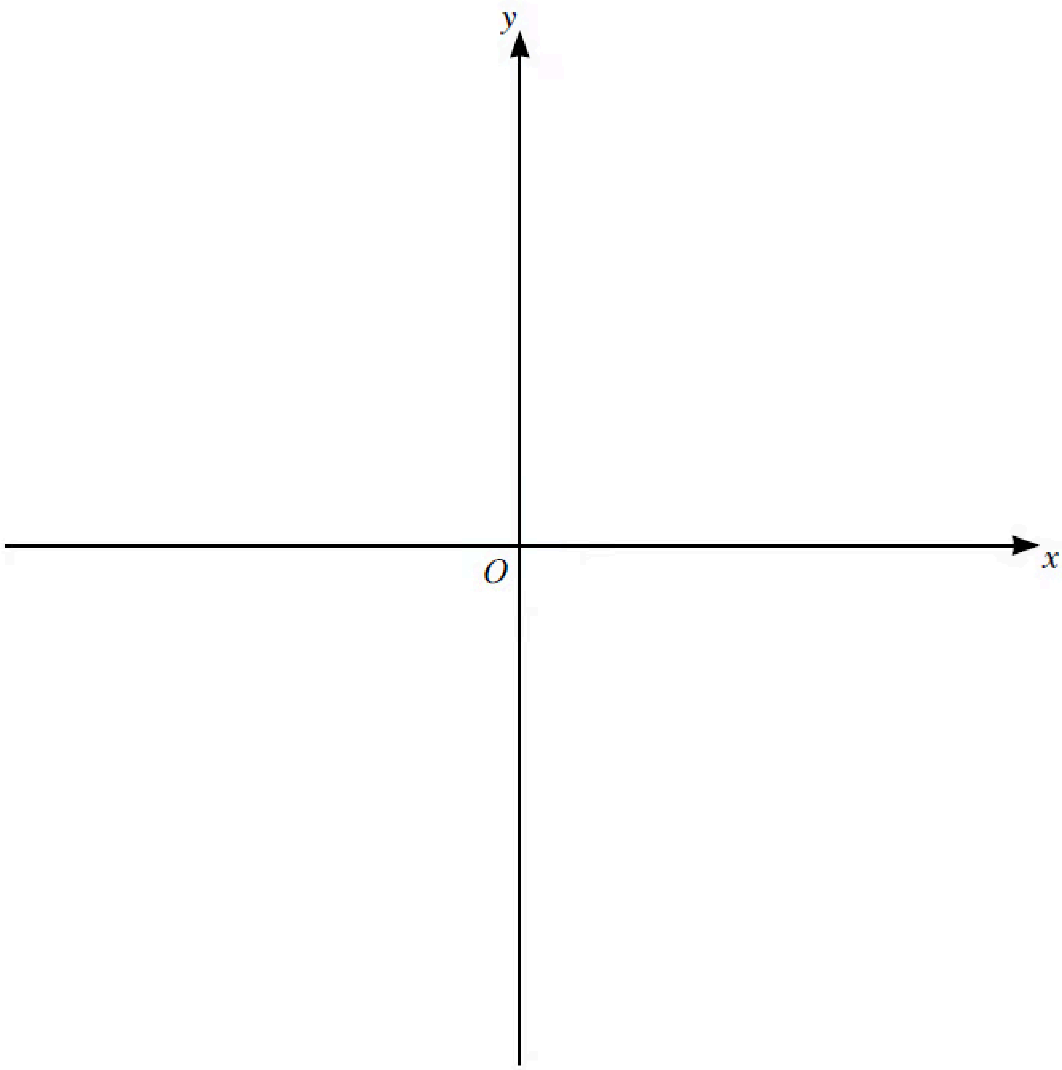
1 $f(x) = 3e^{2x} + 1$ for $x \in \mathbb{R}$
$g(x) = x + 1$ for $x \in \mathbb{R}$

(i) Write down the range of f and the range of g.

(ii) Find $g^2(0)$.

(iii) Hence find $fg^2(0)$.

(iv) On the axes below, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. State the intercepts with the coordinate axes and the equations of any asymptotes.



Answer

i) $ae^{bx} > 0$ therefore

$$3e^{2x} > 0$$

Therefore

$$3e^{2x} + 1 > 1$$

Therefore the range of $f(x)$ is

$$\mathbf{f(x) > 1}\ [1]$$

$g(x) = x + 1$ is a linear function therefore there is no restriction on the range

$$\mathbf{g(x) \in \mathbb{R}}\ [1]$$

ii) Method 1

Find $g(0)$

$$g(0) = 0 + 1 = 1$$

Find $g(1)$

$$g(1) = 1 + 1 = 2$$

Method 2

Find $g^2(x)$

$$g^2(x) = g \circ g(x) = (x + 1) + 1 = x + 2$$

Now find $g^2(0)$

$$g^2(0) = 0 + 2 = 2$$

Either method leads to

$$\mathbf{g^2(0) = 2}\ [1]$$

iii) Find $f(2)$

$$f(2) = 3e^{2 \times 2} + 1$$

[1]

Evaluate

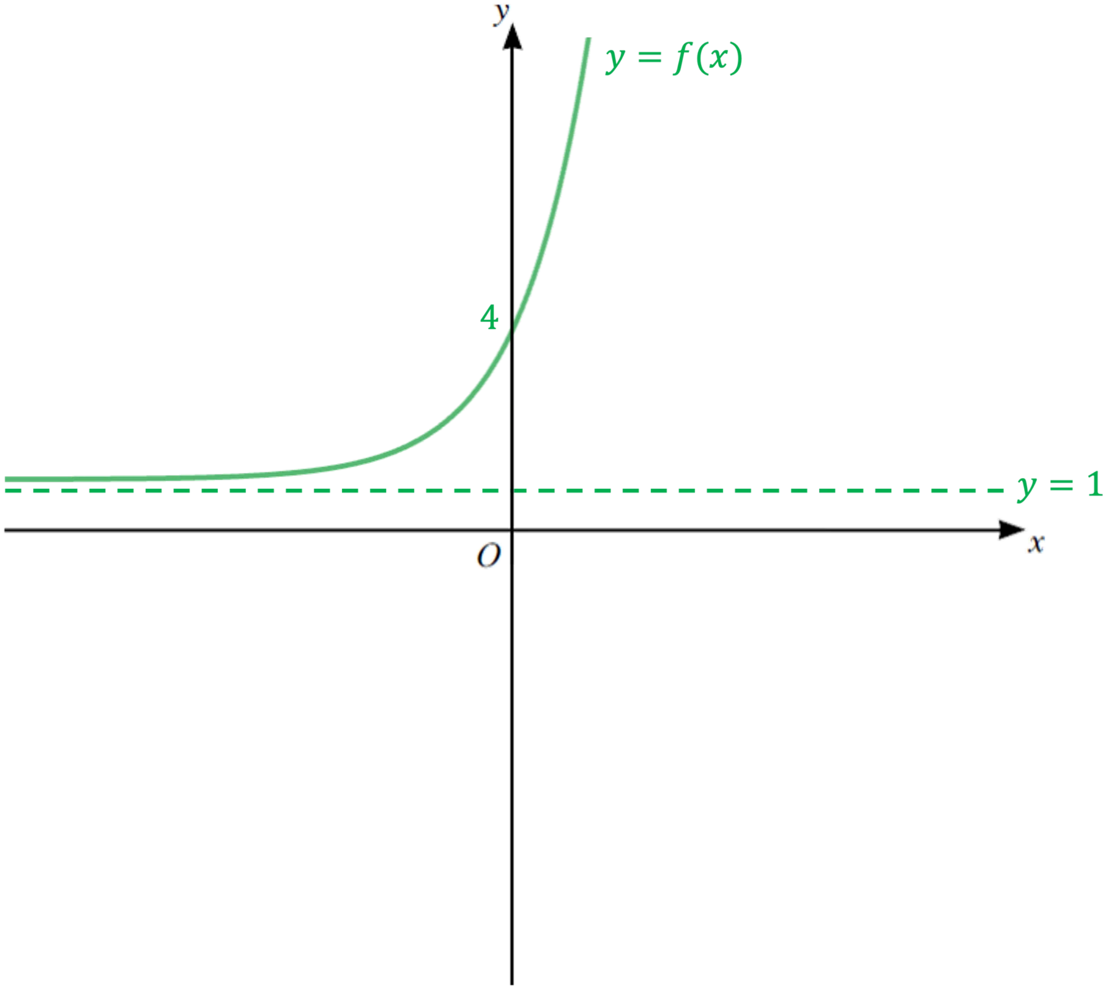
$f(2) = 3e^4 + 1$ [1]

iv) Find $f(0)$ to be sure of the y-intercept

$$f(0) = 3e^{2 \times 0} + 1 = 3 + 1 = 4$$

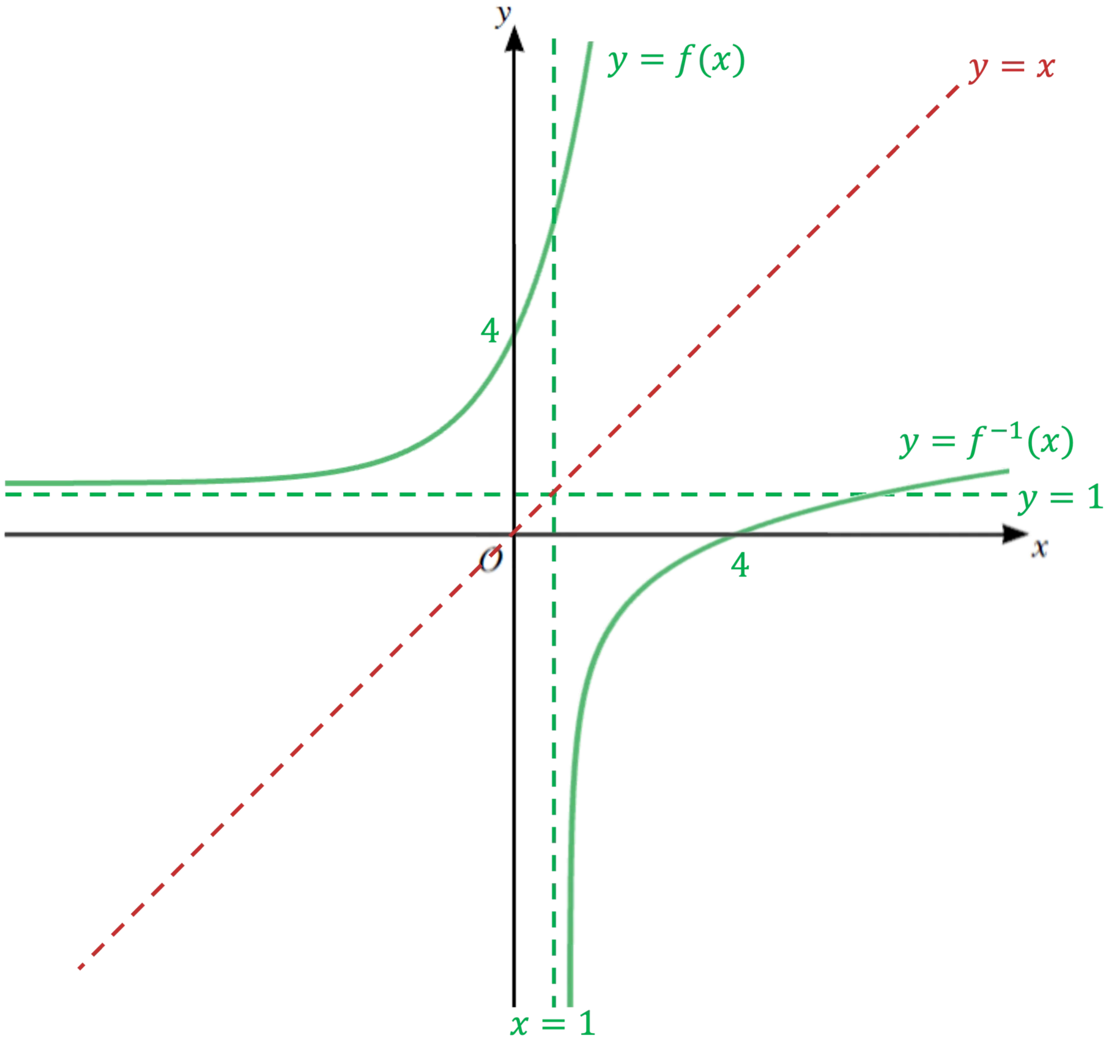
So the curve crosses the y-axis at 4. In addition we know from part (i) that $f(x) > 1$ therefore there is a horizontal asymptote at $y = 1$ and we should know the general shape of the graph of an exponential function. So we are ready to sketch $y = f(x)$.

Make sure the the curve passes through (0, 4) and mark the horizontal asymptote with a broken line and its equation



[1]

$y = \mathrm{f}^{-1}(x)$ is a reflection of $y = \mathrm{f}(x)$ in $y = x$. Draw $y = x$ on the diagram and sketch a reflection of $y = \mathrm{f}(x)$ on it Note that the reflection should go through (4, 0) and there will be a vertical asymptote at $x = 1$



correct shape of inverse [1]

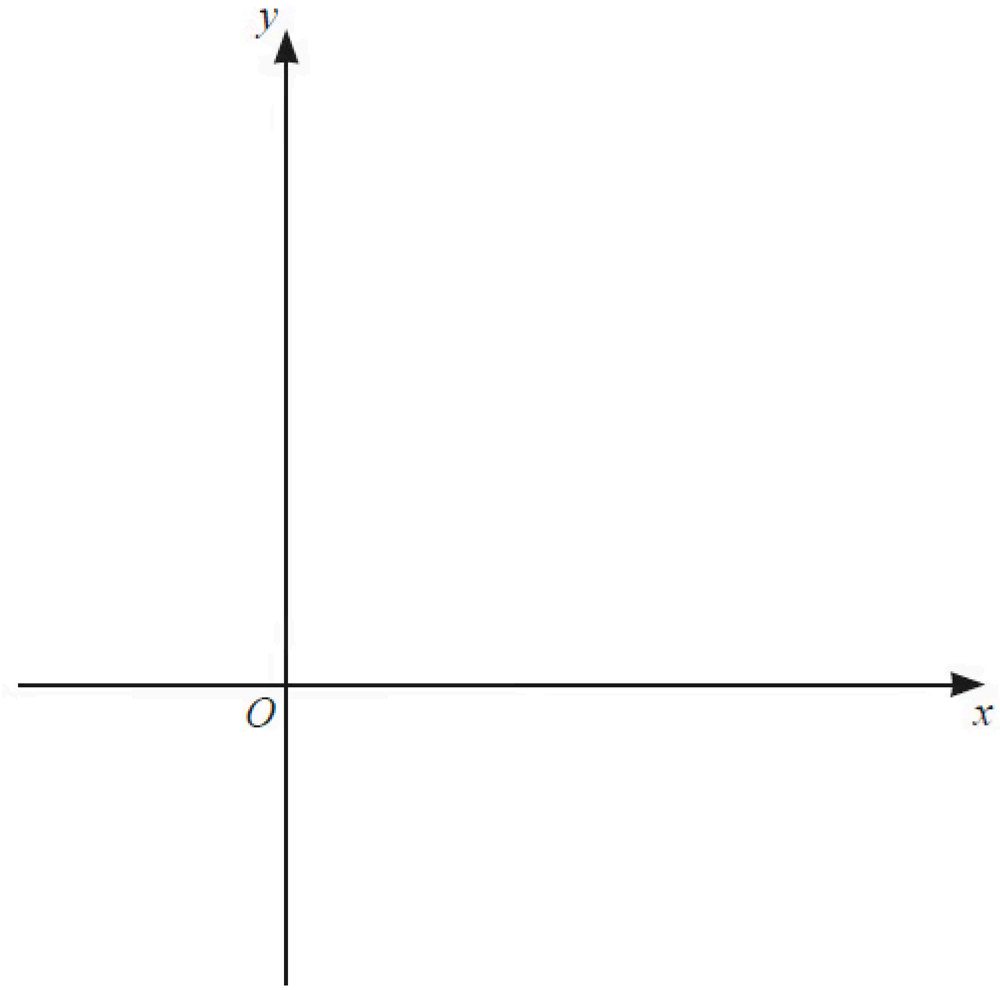
correct asymptotes [1]

correct symmetry in y = x [1]

(9 marks)

2 (a) The function f is defined by $\mathrm{f}(x) = \ln(2x + 1)$ for $x \geq 0$.

Sketch the graph of $y = f(x)$ and hence sketch the graph of $y = f^{-1}(x)$ on the axes below.

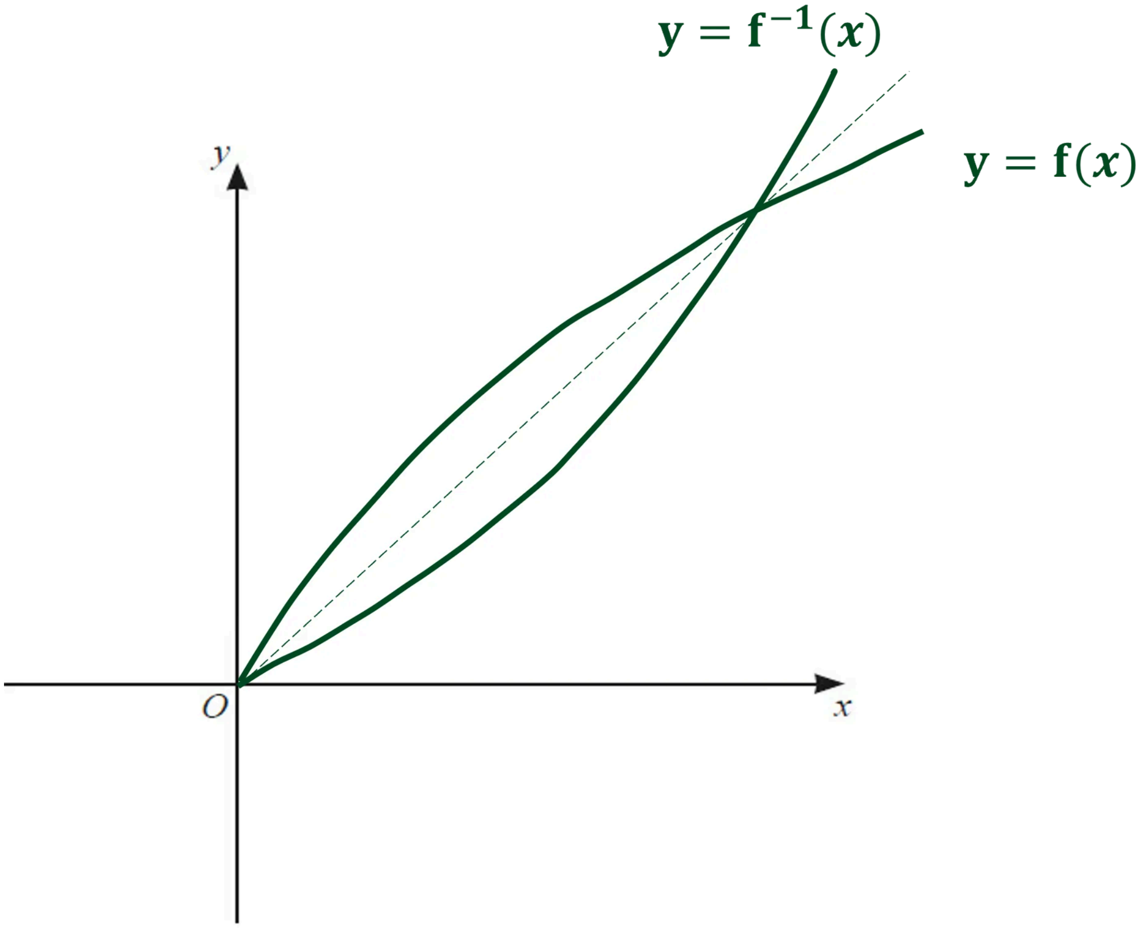


Answer

The key features of the logarithmic graph $y = \ln x$ are that it has an asymptote at $x = 0$, a root at $(1, 0)$ and does not have any minimum or maximum points.

The function $f(x) = \ln(2x + 1)$ has been translated by 1 unit to the left, then stretched in the horizontal direction by a scale factor $\frac{1}{2}$, so now passes through $(0, 0)$. The questions wants a sketch over the domain $x \geq 0$.

The graph of the inverse function is a reflection in the line $y = x$.



correct shape of f or f^{-1} [1]

for symmetry of f and f^{-1} in the line y = x [1]

sketched over the correct domain, x ≥ 0 [1]

(3 marks)

(b) The function g is defined by $g(x) = (x-4)^2 + 1$ for $x \leq 4$.

(i) Find an expression for $g^{-1}(x)$ and state its domain and range.

[4]

(ii) Find and simplify an expression for $fg(x)$.

[2]

(iii) Explain why the function gf does not exist.

[1]

Answer

i) To find the inverse function, set the equation equal to y and rearrange to make x the subject.

$$y = (x - 4)^2 + 1$$

$$y - 1 = (x - 4)^2$$

$$\pm\sqrt{y - 1} = x - 4$$

$$\pm\sqrt{y - 1} + 4 = x$$

[1]

Since the domain of g is given as $x \leq 4$, we disregard the positive square root. The domain of the inverse is the same as the range of the original function and vice versa.

$\mathrm{g}^{-1}(x) = 4 - \sqrt{x - 1}$ [1]

$\mathrm{g}^{-1} \leq 4$ [1]

$x \geq 1$ [1]

ii) To find the composite function $\mathrm{fg}(x)$, substitute the equation for $\mathrm{g}(x)$ into the equation for $\mathrm{f}(x)$

$$\mathrm{fg}(x) = \ln(2\,[(x - 4)^2 + 1] + 1)$$

[1]

Expanding the brackets gives

$$\mathrm{fg}(x) = \ln(2\,[x^2 - 8x + 16 + 1] + 1)$$

Simplifying

$\mathrm{fg}(x) = \ln(2x^2 - 16x + 35)$ [1]

iii)

The function gf does not exist because the range of $f(x)$ includes values that are not valid in the domain of $g(x)$ [1]

(7 marks)

3 (a)

$$f(x) = 3 + e^x \text{ for } x \in \mathbb{R}$$
$$g(x) = 9x - 5 \text{ for } x \in \mathbb{R}$$

Find the range of f and of g.

Answer

For e^x, as $x \to -\infty$, $e^x \to 0$

e^x is greater than 0, and will never reach 0, and therefore the function must be greater than 3.

$f > 3$ [1]

The function $g(x)$ is a linear function, and its domain is x is all real numbers. Therefore, its range is all real numbers $(-\infty, \infty)$.

$g \in \mathbb{R}$ [1]

(2 marks)

(b) Find the exact solution of $f^{-1}(x) = g'(x)$.

Answer

Work out the inverse of $f(x)$.

$$y = 3 + e^x$$
$$y - 3 = e^x$$
$$\ln(y - 3) = x$$

Switch x and y to find the inverse function.

$$y = \ln(x - 3)$$

$$f^{-1}(x) = \ln(x - 3)$$

[1]

Differentiate $g(x)$.

$$g'(x) = 9$$

Put $f^{-1}(x)$ and $g'(x)$ equal to each other, exponentiate both sides and solve.

$$\ln(x-3)=9$$
$$x-3=e^9$$

[1]

$x=e^9+3$ [1]

(3 marks)

(c) Find the solution of $g^2(x) = 112$.

Answer

$g^2(x) = gg(x)$

Substitute $g(x)$ into $g(x)$ and put equal to 112.

$$9(9x-5)-5=112$$
$$81x-50=112$$

[1]

Solve.

$$81x=162$$

$x=2$ [1]

(2 marks)