IGCSE · Cambridge (CIE) · Further Maths

2 hours · 24 questions

Exam Questions

Logarithmic & Exponential Functions

Exponential Functions / Logarithmic Functions / Laws of Logarithms / Exponential Equations / Transforming Relationships to Linear Form

Medium (7 questions)	/25
Hard (8 questions)	/44
Very Hard (9 questions)	/47
Total Marks	**/116**

Medium Questions

1 Given that $\log_a p + \log_a 5 - \log_a 4 = \log_a 20$, find the value of p.

Answer

All the logs have the same base, a

Use the laws of logarithms to write the three terms on the left hand side as one term

$$\log_a p + \log_a 5 - \log_a 4 = \log_a 20$$
$$\log_a (5p) - \log_a 4 = \log_a 20$$
$$\log_a \left(\frac{5p}{4}\right) = \log_a 20$$

[1]

Therefore we can eliminate the logs to leave

$$\frac{5p}{4} = 20$$

Now solve for p (Multiply both sides by 4)

$$5p = 80$$

(Divide both sides by 5)

$p = 16$ [1]

(2 marks)

2 In this question, a, b, c and d are positive constants.

(i) It is given that $y = \log_a(x+3) + \log_a(2x-1)$. Explain why x must be greater than $\frac{1}{2}$.

(ii) Find the exact solution of the equation $\dfrac{\log_a 6}{\log_a(y+3)} = 2$

Answer

i) Since we cannot take the logarithm of a negative number, both $(x+3)$ and $(2x-1)$ must be greater than 0. If $2x-1>0$ then $x>\frac{1}{2}$.

This ensures the argument of both logarithms is greater than 0 [1]

ii) Multiply both sides by $\log_a(y+3)$

$$\log_a 6 = 2\log_a(y+3)$$

Use the logarithm power rule to rewrite.

$$\log_a 6 = \log_a(y+3)^2$$

[1]

Remove logarithms.

$$6 = (y+3)^2$$

[1]

Solve.

$$\pm\sqrt{6} = y+3$$

$$y = -3 \pm \sqrt{6}$$

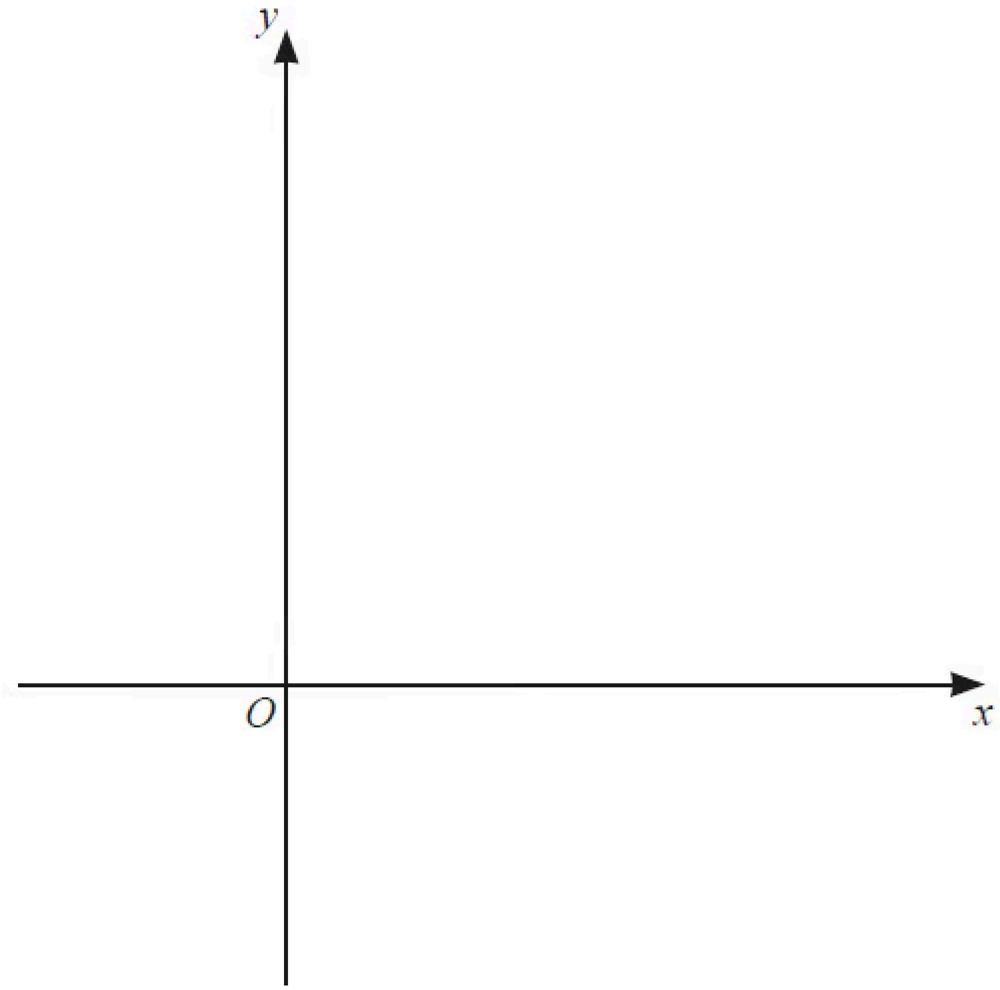
$(y+3) \geq 0$, therefore y cannot equal $-3-\sqrt{6}$

$\mathbf{y = -3+\sqrt{6}}$ [1]

(4 marks)

3 The function f is defined by $f(x) = \ln(2x+1)$ for $x \geq 0$.

Sketch the graph of $y = \mathrm{f}(x)$ and hence sketch the graph of $y = \mathrm{f}^{-1}(x)$ on the axes below.

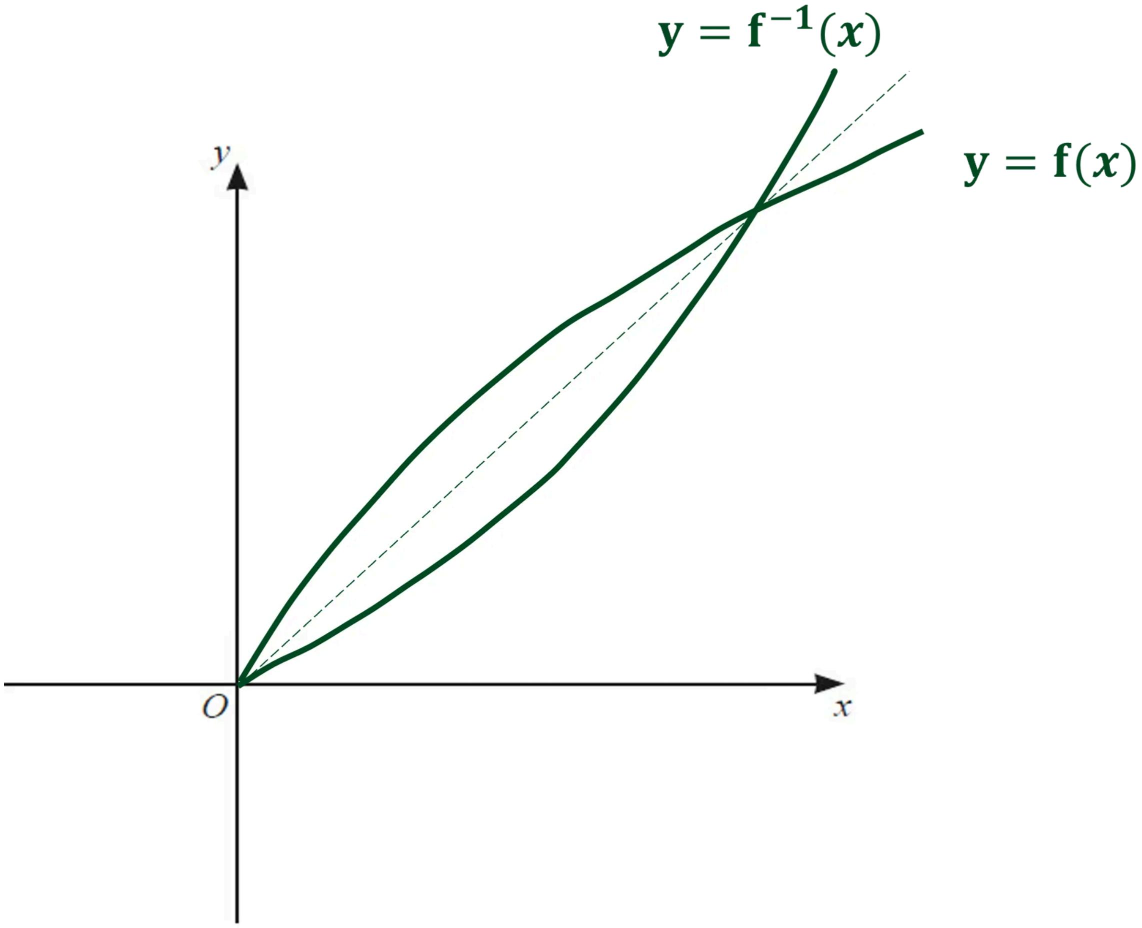


Answer

The key features of the logarithmic graph $y = \ln x$ are that it has an asymptote at $x = 0$, a root at $(1, 0)$ and does not have any minimum or maximum points.

The function $\mathrm{f}(x) = \ln(2x + 1)$ has been translated by 1 unit to the left, then stretched in the horizontal direction by a scale factor $\frac{1}{2}$, so now passes through $(0, 0)$. The questions wants a sketch over the domain $x \geq 0$.

The graph of the inverse function is a reflection in the line $y = x$.



correct shape of f or f^{-1} [1]

for symmetry of f and f^{-1} in the line y = x [1]

sketched over the correct domain, $x \geq 0$ [1]
(3 marks)

4 Solve the equation $\dfrac{9^{5x}}{27^{x-2}} = 243$

Answer

Multiply both sides by 27^{x-2}

$$9^{5x} = 243 \times 27^{x-2}$$

Write each term as a power of 3 (change the base to 3)

$$(3^2)^{5x} = 3^5 \times (3^3)^{x-2}$$

Simplify each term using index laws

$$3^{10x} = 3^5 \times 3^{3x-6}$$

Divide both sides by 3^5

$$\frac{3^{10x}}{3^5} = 3^{3x-6}$$

[1]

Simplify the left hand side

$$3^{10x-5} = 3^{3x-6}$$

[1]

The powers on each side of the equation must be equal ($3^a = 3^b$ means $a = b$) so

$$10x - 5 = 3x - 6$$

Solve the equation to find x

$$7x - 5 = -6$$

$$7x = -1$$

$$x = -\frac{1}{7}$$

$x = -\frac{1}{7}$ [1]

(3 marks)

5 Write $3\lg x + 2 - \lg y$ as a single logarithm.

Answer

In order to apply log laws, each term must be in terms of logs. We can write 2 as $\lg 100$ because $10^2 = 100$ and $\lg$ means base 10. Therefore we have

$$3\lg x + \lg 100 - \lg y$$

[1]

Applying the log law $a\log b = \log b^a$

$$\lg x^3 + \lg 100 - \lg y$$

Applying the log law $\log a + \log b = \log ab$

$$\lg 100x^3 - \lg y$$

[1]

Applying the log law $\log a - \log b = \log\left(\frac{a}{b}\right)$

$\lg\left(\frac{100x^3}{y}\right)$ [1]

(3 marks)

6 Using the substitution $y = 2^x$, or otherwise, solve $2^{2x+1} - 2^{x+1} - 2^x + 1 = 0$.

Answer

Rewrite the question using index laws so the powers are 2^x

$$(2^1)(2^{2x}) - (2^1)(2^x) - (2^x) + 1 = 0$$
$$(2)(2^x)^2 - (2)(2^x) - (2^x) + 1 = 0$$

Substitute $y = 2^x$ into the equation.

$$2y^2 - 2y - y + 1 = 0$$
$$2y^2 - 3y + 1 = 0$$

[1]

Factorise.

$$(2y-1)(y-1)=0$$

Solve for y.

$$y=\frac{1}{2} \text{ and } y=1$$

[1]

Substitute $y=2^x$ into the solutions.

$$2^x=\frac{1}{2} \text{ and } 2^x=1$$

Using index laws, solve for x.

$\boldsymbol{x=-1}$ [1]

$\boldsymbol{x=0}$ [1]

(4 marks)

7 (a) Solve the following equation.

$$\frac{1+3e^x}{e^x+e^{-x}}=1$$

Give your answer(s) in the form $p\ln q$ where p and q are integers.

Answer

Multiply both sides by the denominator

$$1+3e^x=e^x+e^{-x}$$

This is a hidden quadratic in e^x so let $u=e^x$

This means that $e^{-x}=\frac{1}{e^x}=\frac{1}{u}$

$$1+3u=u+\frac{1}{u}$$

[M1]

Mark Scheme and Guidance

This mark is for recognising that it leads to a quadratic and attempting to write the quadratic in the form $a(\ )^2 + b(\ ) + c = 0$.

Rearrange into a quadratic equation (start by multiplying both sides by u)

$$u(1+3u) = u^2 + 1$$

Expand and bring terms to one side

$$u + 3u^2 = u^2 + 1$$
$$2u^2 + u - 1 = 0$$

[A1]

Solve the quadratic equation, e.g. by factorisation

$$(2u-1)(u+1) = 0$$

[M1]

$$u = \frac{1}{2} \text{ or } u = -1$$

Substitute $u = e^x$ back into the answers

$$e^x = \frac{1}{2} \text{ or } e^x = -1$$

The graph $y = e^x$ is always positive so there are no solutions to $e^x = -1$

Solve $e^x = \frac{1}{2}$ by taking natural logarithms

$$x = \ln\left(\frac{1}{2}\right)$$

Write this in the form $p\ln q$ by using log laws

$$x = \ln(2^{-1}) = -\ln 2$$

$x = -\ln 2$

[A1]

(4 marks)

(b) Given that $\log_2 y = c$, express $\log_2(2y) - \log_y 2$ in terms of c only.

Answer

Use the log law $\log(ab) = \log a + \log b$ for the first term

$$\log_2(2y) = \log_2 2 + \log_2 y$$

Use that $\log_2 2 = 1$ and that $\log_2 y = c$

$$\log_2(2y) = 1 + c$$

For the second term, apply a change of base using the rule $\log_a b = \dfrac{1}{\log_b a}$

$$\log_y 2 = \frac{1}{\log_2 y} = \frac{1}{c}$$

Combine the results above

$\log_2(2y) - \log_y 2 = 1 + c - \dfrac{1}{c}$

[B1 B1]

Mark Scheme and Guidance

B1: For having at least $1 + c$ or $-\dfrac{1}{c}$ in your answer, or seeing $\log_y 2 = \dfrac{1}{c}$.

B1: For having a fully correct final answer in any equivalent form, e.g. $\dfrac{c + c^2 - 1}{c}$.

(2 marks)

Hard Questions

1 For variables x and y, plotting $\ln y$ against $\ln x$ gives a straight-line graph passing through the points $(6, 5)$ and $(8, 9)$.
Show that $y = e^p x^q$ where p and q are integers to be found.

Answer

Find the gradient using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{9-5}{8-6} = 2$$

[1]

Substitute one of the points and the gradient into $Y = mX + c$ and solve to find c

$$5 = 2(6) + c$$

[1]

$$c = -7$$

The equation of a straight line in terms of $\ln y$ and $\ln x$ is $\ln y = m\ln x + c$

$$\ln y = 2\ln x - 7$$

[1]

Rewrite $\ln y$ as a power of e

$$y = e^{2\ln x - 7}$$

And rearrange using laws of logs and exponents. Remember that $e^{\ln x} = x$

$$\begin{aligned} y &= e^{\ln x^2 - 7} \\ &= e^{\ln x^2} e^{-7} \\ &= x^2 e^{-7} \end{aligned}$$

$y = e^{-7}x^2$ [1]

(4 marks)

2 Variables x and y are such that, when $\lg y$ is plotted against x^3, a straight line graph passing through the points (6,7) and (10,9) is obtained. Find y as a function of x.

Answer

Calculate the gradient between (6, 7) and (10, 9)

$$\text{gradient} = \frac{9-7}{10-6}$$

$$\text{gradient} = \frac{1}{2}$$

[1]

Calculate the y–intercept of the line using the gradient and the point (6, 7)

$$7 = \left(\frac{1}{2} \times 6\right) + c$$

$$c = 4$$

[1]

Using $y = \mathrm{m}x + \mathrm{c}$ the equation will be

$$\lg y = \frac{1}{2}x^3 + 4$$

[1]

Taking the exponential function of both sides we get

$y = 10^{\frac{1}{2}x^3 + 4}$ [1]

(4 marks)

3 Given that $\log_2 x + 2\log_4 y = 8$, find the value of xy.

Answer

Change the base of $2\log_4 y$ so it has a base of 2 like the other term in the equation.

$$\log_4 y = \frac{\log_2 y}{\log_2 4}$$

$$\log_4 y = \frac{\log_2 y}{2}$$

Therefore,

$$2\log_4 y = \log_2 y$$

[1]

Substitute into the given equation.

$$\log_2 x + \log_2 y = 8$$

Use logarithms laws to combine the left hand side.

$$\log_2 (xy) = 8$$

[1]

Rearrange and solve for xy.

$$xy = 2^8$$

$xy = 256$ [1]

(3 marks)

4 (a) Variables x and y are connected by the relationship $y = Ax^n$, where A and n are constants.

Transform the relationship $y = Ax^n$ to straight line form.

Answer

Take the natural logarithm of both sides.

$$\ln y = \ln(Ax^n)$$

Use logarithm laws to rewrite.

$$\ln y = \ln A + \ln x^n$$

[1]

Use the logarithm power rule to rewrite in the form $Y = mX + c$

$\ln y = n \ln x + \ln A$ [1]

(2 marks)

(b) When $\ln y$ is plotted against $\ln x$ a straight line graph passing through the points (0, 0.5) and (3.2, 1.7) is obtained.

Find the value of n and of A.

Answer

The equation from part a is in the form $Y = mX + c$, therefore the gradient is n and the y intercept is $\ln A$.

The coordinate $(0, 0.5)$ tells us that the y intercept of the line is 0.5.

Therefore,

$$\ln A = 0.5$$

[1]

Exponentiate both sides and simplify.

$$e^{\ln A} = e^{0.5}$$
$$A = e^{0.5}$$

$A = e^{0.5}$ [1]

$A = 1.65$ to 3sf is also accepted

To find the gradient of the line when we know two coordinates, $\frac{y_2 - y_1}{x_2 - x_1}$. This is equal to n.

$$n = \frac{1.7 - 0.5}{3.2 - 0}$$

[1]

$n = \frac{3}{8}$ [1]

(4 marks)

(c) Find the value of y when $x = 11$.

Answer

Substitute $A = e^{0.5}$ and $n = \frac{3}{8}$ into $y = Ax^n$.

$$y = x^{\frac{3}{8}} \times e^{0.5}$$

Substitute the value of x into the equation to find the value of y.

$$y = (11)^{\frac{3}{8}} \times e^{0.5} = 4.0520...$$

[1]

$y = 4.05$ [1]

(2 marks)

5 (a) The population P, in millions, of a country is given by $P = A \times b^t$, where t is the number of years after January 2000 and A and b are constants. In January 2010 the population was 40 million and had increased to 45 million by January 2013.

Show that b = 1.04 to 2 decimal places and find A to the nearest integer.

Answer

In 2010, 10 years have passed, therefore $t = 10$ and we are given $P = 40$.

In 2013, 13 years have passed, therefore $t = 13$ and we are given $P = 45$.

This means that we can form two equations

$$40 = A \times b^{10}$$

$$45 = A \times b^{13}$$

[1]

Solving these simultaneously by dividing the second equation by the first gives

$$\frac{45}{40} = b^3$$

[1]

Taking the cube root of both sides gives

$$1.04004... = b$$

Rounding this to two decimal places, as required, gives

$$1.04 = b$$

[1]

Substituting this in to the first equation

$$40 = A \times (1.04...)^{10}$$

Dividing both sides by 1.04^{10} and rounding to the nearest integer gives

$A = 27$ [1]

(4 marks)

(b) Find the population in January 2020, giving your answer to the nearest million.

Answer

In 2020, 20 years have passed so $t = 20$. Substituting this and the rounded answers to part (a) we have

$$P = 27 \times 1.04^{20}$$

$P = 59$ [1]

(1 mark)

(c) In January of which year will the population be over 100 million for the first time?

Answer

We can use the information to form and solve an inequality in t

$$27 \times 1.04^{t} > 100$$

[1]

Dividing both sides by 27

$$1.04^{t} > \frac{100}{27}$$

Taking logs of both sides

$$\log 1.04^{t} > \log \frac{100}{27}$$

Applying the log law $\log a^{b} = b\log a$

$$t\log 1.04 > \log \frac{100}{27}$$

Dividing both sides by $\log 1.04$

$$t > 33.38...$$

[1]

We know that t represents the number of years after 2000, therefore it will be 34 years before the population exceeds 100 million for the first time if the data is collected in January only.

2034 [1]

(3 marks)

6 (a) The number, b, of bacteria in a sample is given by $b = P + Qe^{2t}$, where P and Q are constants and t is time in weeks. Initially there are 500 bacteria which increase to 600 after 1 week.

Find the value of P and of Q.

Answer

When $t = 0,\ b = 500$

Substitute the above values into the equation

$$500 = P + Qe^{2(0)}$$

Simplify

$$(1) \quad 500 = P + Q$$

When $t = 1,\ b = 600$

Substitute the above values into the equation

$$600 = P + Qe^{2(1)}$$

Simplify

$$(2) \quad 600 = P + Qe^2$$

[1]

Solve equations (1) and (2) simultaneously by subtracting (1) from (2)

$$100 = Qe^2 - Q$$

[1]

Factorise out Q

$$100 = Q(e^2 - 1)$$

Divide both sides by $(e^2 - 1)$

$$Q = \frac{100}{e^2 - 1} = 15.651... = 15.7\ (3sf)$$

[1]

Find P by substituting Q into equation (1)

$$P = 500 - 15.65... = 484.34... = 484\ (3sf)$$

$P = 484,\ Q = 15.7$ [1]

(4 marks)

(b) Find the number of bacteria present after 2 weeks.

Answer

From part (a), the model is $b = 484.3... + 15.65e^{2t}$

When 2 weeks has passed, $t = 2$

Substitute $t = 2$ into the equation to find b

$$b = 484.34... + 15.65... \times e^{2(2)} = 1338.90...$$

1338 [1]

(1 mark)

(c) Find the first week in which the number of bacteria is greater than 1 000 000.

Answer

Make an inequality for when the bacteria is greater than 1 000 000

$$484.3... + 15.65...e^{2t} > 1\ 000\ 000$$

Solve for t

$$15.65...e^{2t} > 999\ 515.7$$

$$e^{2t} > \frac{999\ 515.7}{15.65...}$$

[1]

Take natural logs of both sides

$$2t > \ln\left(\frac{999\ 515.7}{15.65...}\right)$$

[1]

Divide by 2

$$t > \frac{1}{2}\ln\left(\frac{999\ 515.7}{15.65...}\right)$$

$$t > 5.532...$$

Find the first integer which is greater than 5.532...

6th week [1]

(3 marks)

7

$$\mathrm{f}(x) = 3 + \mathrm{e}^x \text{ for } x \in \mathbb{R}$$
$$\mathrm{g}(x) = 9x - 5 \text{ for } x \in \mathbb{R}$$

Find the exact solution of $\mathrm{f}^{-1}(x) = \mathrm{g}'(x)$.

Answer

Work out the inverse of $\mathrm{f}(x)$.

$$y = 3 + \mathrm{e}^x$$
$$y - 3 = \mathrm{e}^x$$
$$\ln(y-3) = x$$

Switch x and y to find the inverse function.

$$y = \ln(x-3)$$

$$\mathrm{f}^{-1}(x) = \ln(x-3)$$

[1]

Differentiate $\mathrm{g}(x)$.

$$\mathrm{g}'(x) = 9$$

Put $\mathrm{f}^{-1}(x)$ and $\mathrm{g}'(x)$ equal to each other, exponentiate both sides and solve.

$$\ln(x-3) = 9$$
$$x - 3 = \mathrm{e}^9$$

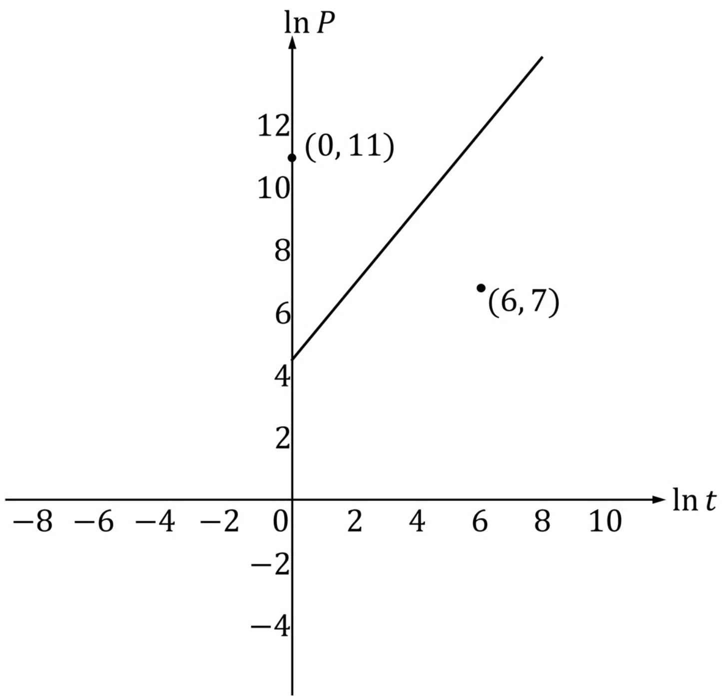
[1]

$x = \mathrm{e}^9 + 3$ [1]

(3 marks)

8 A mathematical model is used to predict the population of an island, P, over time, t years.

A plot of $\ln P$ against $\ln t$ gives a straight line that is the **perpendicular bisector** of the points $(0,\ 11)$ and $(6,\ 7)$, as shown.



Find and simplify a formula for P in terms of t.

Answer

Find the midpoint of $(0,\ 11)$ and $(6,\ 7)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)=\left(\frac{0+6}{2}, \frac{11+7}{2}\right)$$
$$=\left(\frac{6}{2}, \frac{18}{2}\right)$$
$$=(3, 9)$$

[B1]

Find the gradient of the line from $(0, 11)$ to $(6, 7)$

$$\frac{7-11}{6-0}=-\frac{4}{6}$$
$$=-\frac{2}{3}$$

[M1]

Find the perpendicular gradient (the negative reciprocal)

$$\frac{3}{2}$$

[A1]

Find the equation of the perpendicular bisector (the line through $(3, 9)$ with gradient $\frac{3}{2}$
)

$$y-y_1=m(x-x_1)$$
$$y-9=\frac{3}{2}(x-3)$$
$$y-9=\frac{3}{2}x-\frac{9}{2}$$
$$y=\frac{3}{2}x+\frac{9}{2}$$

[A1]

Examiner Tips and Tricks

A quick check of the graph confirms that this line meant to have a positive gradient and a y-intercept between 4 and 6.

Substitute in $y = \ln P$ and $x = \ln t$

$$\ln P = \frac{3}{2}\ln t + \frac{9}{2}$$

[M1]

Make P the subject by raising both sides to the power e

$$P = e^{\frac{3}{2}\ln t + \frac{9}{2}}$$

Simplify the answer using index and log laws (and that $e^{\ln \cdots}$ cancel out)

$$P = e^{\frac{3}{2}\ln t} e^{\frac{9}{2}}$$
$$P = e^{\ln t^{\frac{3}{2}}} e^{\frac{9}{2}}$$
$$P = t^{\frac{3}{2}} e^{\frac{9}{2}}$$
$$P = e^{\frac{9}{2}} t^{\frac{3}{2}}$$

$P = e^{\frac{9}{2}} t^{\frac{3}{2}}$

[A1]

Mark Scheme and Guidance

Answers which round to $P = 90.0t^{\frac{3}{2}}$ are also accepted.

(6 marks)

Very Hard Questions

1 Solve the equation $3^{2x+1} + 8(3^x) - 3 = 0$.

Answer

The trick here is to recognise that $3^{2x+1} = 3^{2x} \times 3^1 = (3^x)^2 \times 3$, so we could rewrite '$3^{2x-1} + 8(3^x) - 3 = 0$' as

$$3(3^x)^2 + 8(3^x) - 3 = 0$$

Now we can recognise that we have a quadratic equation in '3^x'. If you like, replace '3^x' with a letter, like y

$$3y^2 + 8y - 3 = 0$$

Factorise this

$$(3y - 1)(y + 3) = 0$$

[1]

And solve to find y (or 3^x)

$$y = \frac{1}{3}, y = -3$$

Remember that a number to the power of something can't be negative therefore

$$3^x = \frac{1}{3}, 3^x \neq -3$$

[1]

If necessary, rewrite $\frac{1}{3}$ as a power of 3

$$3^x = 3^{-1}$$

$\boldsymbol{x = -1}$ [1]

(3 marks)

2 Solve the equation $4 \log_y 2 + \log_2 y = 4$.

Answer

We have two logs with different bases so we need to change them to the same base. Here we will change them both to $\log_2$ but you could also change to $\log_y$ and then use a similar method with a quadratic equation in $\log_y$

If you don't recognise $\log_2 y$ as $1/(\log_y 2)$ then use the change of base rule,

$$\log_x y = \frac{\log_a y}{\log_a x}$$

$$\log_y 2 = \frac{\log_2 2}{\log_2 y} = \frac{1}{\log_2 y}$$

or writing $\log_2 y$ *as* $\frac{1}{\log_y 2}$ [1]

Now rewrite the equation in terms of $\log_2$

$$\frac{4}{\log_2 y} + \log_2 y = 4$$

Multiply all terms by $\log_2 y$ to form a quadratic equation in $\log_2 y$

$$4 + (\log_2 y)^2 = 4\log_2 y$$

If you like, replace $\log_2 y$ with a letter, like x, and rearrange to an equation equalling 0

$$4 + x^2 = 4x$$
$$x^2 - 4x + 4 = 0$$

Factorise and solve

$$x^2 - 4x + 4 = 0$$
$$(x - 2)^2 = 0$$
$$x = 2$$

Remember that $x = \log_2 y$ therefore

$$\log_2 y = 2$$

or " $\log_y 2 = \frac{1}{2}$ *" using similar method* [1]

Rewrite in exponent form

$$2^2 = y$$

$y = 4$ [1]

(3 marks)

3 Write the expression $\log_a 9 + (\log_a b)(\log_{\sqrt{b}} 9a)$ in the form $c + d\log_a 9$, where c and d are integers.

Answer

Change the base of $\log_{\sqrt{b}} 9a$ to base a.

$$\frac{\log_a 9a}{\log_a \sqrt{b}}$$

[1]

Use the logarithm power law to rewrite the denominator.

$$\frac{\log_a 9a}{\left(\frac{1}{2}\right)\log_a b}$$

Use the addition/multiplication law to rewrite the numerator.

$$\frac{\log_a 9 + \log_a a}{\left(\frac{1}{2}\right)\log_a b}$$

[1]

$\log_a a = 1$

$$\frac{(\log_a 9) + 1}{\left(\frac{1}{2}\right)\log_a b}$$

[1]

Substitute into original equation.

$$\log_a 9 + (\log_a b)\left(\frac{(\log_a 9) + 1}{\left(\frac{1}{2}\right)\log_a b}\right)$$

Simplify.

$$\log_a 9 + \left(\frac{(\log_a 9) + 1}{\left(\frac{1}{2}\right)}\right)$$

$$\log_a 9 + 2(\log_a 9) + 2$$

$2 + 3\log_a 9$ [1]

(4 marks)

4 Find the exact solution of $3^{2x} - 3^{x+1} - 4 = 0$.

Answer

Let $y = 3^x$ then

$$3^{2x} = 3^x \times 3^x = y^2$$

$$3^{x+1} = 3^x \times 3^1 = 3y$$

Use the original equation and change the variables

$$3^{2x} - 3^{x+1} - 4 = 0$$

$$y^2 - 3y - 4 = 0$$

[1]

Factorise and solve the quadratic equation

$$(y+1)(y-4)=0$$

$$y=-1 \text{ or } y=4$$

[1]

But $y=3^x$ so

$$3^x=-1 \text{ or } 3^x=4$$

[1]

Solve each equation using the inverse which is $\log_3$

$$x=\log_3(-1) \text{ or } x=\log_3 4$$

The first equation has no solutions so disregard this and our final answer is

$\boldsymbol{x=\log_3 4}$ [1]

(4 marks)

5 Solve the simultaneous equations

$$10^{x+2y} = 5,$$

$$10^{3x+4y} = 50,$$

giving x and y in exact simplified form.

Answer

Taking logs of both sides of each equation

$$\lg 10^{x+2y} = \lg 5$$

$$\lg 10^{3x+4y} = \lg 50$$

$\lg 10 = 1$, therefore

$$x + 2y = \lg 5$$

$$3x + 4y = \lg 50$$

[1]

Multiplying the first equation by 2 gives

$$2x + 4y = 2\lg 5$$

Subtracting this from the second equation gives

$$x = \lg 50 - 2\lg 5$$

[1]

Using the log laws that $a\log b = \log b^a$ and $\log m - \log n = \log\frac{m}{n}$

$$x = \lg 50 - \lg 25 = \lg 2$$

[1]

Substituting this into the equation $x + 2y = \lg 5$

$$\lg 2 + 2y = \lg 5$$

$$2y = \lg\frac{5}{2}$$

$x = \lg 2,\ y = \frac{1}{2}\lg\frac{5}{2}$ [1]

(4 marks)

6 $\log_a\sqrt{b} - \frac{1}{2} = \log_b a$, where $a > 0$ and $b > 0$.

Solve this equation for b, giving your answers in terms of a.

Answer

$$\log_a \sqrt{b} - \frac{1}{2} = \log_b a$$

Rewrite the square root as a power

$$\log_a b^{\frac{1}{2}} - \frac{1}{2} = \log_b a$$

Bring down the $\frac{1}{2}$

$$\frac{1}{2}\log_a b - \frac{1}{2} = \log_b a$$

[1]

Change the base of the log on the right hand side

$$\frac{1}{2}\log_a b - \frac{1}{2} = \frac{1}{\log_a b}$$

[1]

Multiply through by 2

$$\log_a b - 1 = \frac{2}{\log_a b}$$

Multiply through by $\log_a b$

$$(\log_a b)^2 - \log_a b = 2$$

Make the equation equal to 0

$$(\log_a b)^2 - \log_a b - 2 = 0$$

[1]

Factorise the quadratic equation

$$\left(\log_a b - 2\right)\left(\log_a b + 1\right) = 0$$

[1]

Find the solutions

Either $\log_a b = 2$ or $\log_a b = -1$

Make b the subject

$\log_a b = 2$ leads to $a^2 = b$ $\log_a b = -1$ leads to $a^{-1} = b$

$\boldsymbol{b = a^2}$ **or** $\boldsymbol{b = \frac{1}{a}}$ [1]

(5 marks)

7 Solve the simultaneous equations.

$$\log_3(x + y) = 2$$

$$2\log_3(x + 1) = \log_3(y + 2)$$

Answer

The first equation is $\log_3(x + y) = 2$

Use the inverse of $\log_3$ which is doing 3 to the power of both sides

$$x + y = 3^2$$

$$(1) \qquad x + y = 9$$

[1]

The second equation is $2\log_3(x + 1) = \log_3(y + 2)$

Move the 2 from in front to be a power

$$\log_3(x + 1)^2 = \log_3(y + 2)$$

Use the inverse of $\log_3$ which is doing 3 to the power of both sides

$$(x+1)^2 = y+2$$

[1]

Subtract 2 from both sides to make y the subject

$$(2) \qquad y = (x+1)^2 - 2$$

Solve equations (1) and (2) simultaneously by substituting equation (2) into equation (1)

$$x + (x+1)^2 - 2 = 9$$

[1]

Expand the double bracket

$$x + x^2 + x + x + 1 - 2 = 9$$

Simplify the left hand side

$$x^2 + 3x - 1 = 9$$

Subtract 9 from both sides to make the quadratic equal 0

$$x^2 + 3x - 10 = 0$$

[1]

Solve by factorising (or using the quadratic formula)

$$(x+5)(x-2) = 0$$

$$x = -5 \text{ or } x = 2$$

[1]

Find y by substituting each value into equation (1)

When $x = -5$

$$y = 9 - (-5) = 14$$

Reject this as we cannot have $\log_3(\text{negative})$

When $x = 2$

$$y = 9 - 2 = 7$$

$x = 2$ and $y = 7$ [1]

(6 marks)

8 (a) It is known that $y = A \times 10^{bx^2}$, where A and b are constants. When $\lg y$ is plotted against x^2, a straight line passing through the points (3.63, 5.25) and (4.83, 6.88) is obtained.

Find the value of A and of b.

Answer

$y = A \times 10^{bx^2}$

Take logarithms of both sides.

$$\lg y = \lg\left(A \times 10^{bx^2}\right)$$

Use logarithm laws to rewrite.

$$\lg y = \lg A + bx^2$$

$$\lg y = bx^2 + \lg A$$

[1]

Find the gradient between the two coordinates.

$$m = \frac{6.88 - 5.25}{4.83 - 3.63}$$

$$m = 1.3583...$$

Substitute $m = 1.36$, $X = x^2$ and $y = \lg(y)$ into the equation of a line $(Y = mX + c)$.

$$\lg y = 1.358x^2 + \lg A$$

Compare to $\lg y = bx^2 + \lg A$

$$b = 1.358$$

Substitute $b = 1.358$ and one of the coordinates into $Y = mX + c$ to find the value of c
.

$$5.25 = 1.358(3.63) + c$$

$$c = 0.32046$$

Compare to $\lg y = bx^2 + \lg A$

$$\lg A = 0.32046$$

Solve by raising 10 to the power of both sides.

$$A = 10^{0.32046}$$
$$A = 2.0915...$$

$\mathbf{A = 2.09}$ [1]

$\mathbf{b = 1.36}$ [1]

(4 marks)

(b) Using your values of A and b, find the value of y when $x = 2$,

Answer

Substitute $A = 2.09$ and $b = 1.36$ into $y = A \times 10^{bx^2}$

$$y = 2.09 \times 10^{1.36x^2}$$

Substitute $x = 2$ into the equation.

$$y = 2.09 \times 10^{(1.36)(2)^2}$$

[1]

Solve for y.

$$y = 575633.799$$

$y = 576000$ [1]

answers in the range 553000 to 576000 are accepted

(2 marks)

(c) Find the positive value of x when $y = 4$.

Answer

Substitute $y = 4$ into $y = 2.09 \times 10^{1.36x^2}$

$$4 = 2.09 \times 10^{1.36x^2}$$

[1]

Rearrange.

$$\frac{4}{2.09} = 10^{1.36x^2}$$

Take logarithms of both sides.

$$\lg\left(\frac{4}{2.09}\right) = 1.36x^2$$

Solve for x.

$$x^2 = \frac{\lg\left(\frac{4}{2.09}\right)}{1.36}$$

$$x = \sqrt{\frac{\lg\left(\frac{4}{2.09}\right)}{1.36}}$$

$x = 0.455$ [1]

answers which round to 0.46 are accepted

(2 marks)

9 (a)

$$\log_2(y+1) = 3 - 2\log_2 x$$

$$\log_2(x+2) = 2 + \log_2 y$$

Show that $x^3 + 6x^2 - 32 = 0$.

Answer

Rearrange the first equation.

$$\log_2(y+1) + 2\log_2 x = 3$$

Rewrite using logarithm laws.

$$\log_2(y+1) + \log_2 x^2 = 3$$

Combine using logarithm laws.

$$\log_2((y+1)(x^2)) = 3$$

Rewrite using logarithm laws.

$$(y+1)(x^2) = 2^3$$
$$(y+1)(x^2) = 8$$

[1]

Rearrange the second equation.

$$\log_2(x+2) - \log_2 y = 2$$

Rewrite using logarithm laws.

$$\log_2\left(\frac{x+2}{y}\right) = 2$$

Rewrite using logarithm laws.

$$\frac{x+2}{y}=2^2$$

$$x+2=4y$$

[1]

Solve the two equations $(y+1)(x^2)=8$ and $x+2=4y$ simultaneously. For example, by rearranging the second equation and using a substitution method.

$$y=\frac{x+2}{4}$$

Substitute into the first equation.

$$(x^2)\left(\frac{x+2}{4}+1\right)=8$$

[1]

Expand the brackets.

$$\frac{(x^2)(x+2)}{4}+x^2=8$$

$$\frac{x^3+2x^2}{4}+x^2=8$$

Multiply through by 4.

$$x^3+2x^2+4x^2=32$$

Simplify and rearrange to write in the form specified in the question.

$$x^3+6x^2=32$$

$x^3+6x^2-32=0$ [1]

(4 marks)

(b) Find the roots of $x^3 + 6x^2 - 32 = 0$.

Answer

$f(x) = x^3 + 6x^2 - 32$

Use factor theorem to identify one of the roots.

Since $f(2) = 0$, one of the roots of the polynomial is:

$$x = 2$$

This means that one of the factors of the polynomial is:

$$(x - 2)$$

[1]

Use algebraic (polynomial) division to find the quadratic factor.

$$\begin{array}{r|l} & x^2 \quad +8x \quad +16 \\ \hline x-2 \;) & x^3 + 6x^2 + 0x - 32 \\ & x^3 - 2x^2 \qquad\qquad - \\ \hline & \qquad 8x^2 - 0x \\ & \qquad 8x^2 - 16x \qquad - \\ \hline & \qquad\qquad 16x - 32 \\ & \qquad\qquad 16x - 32 \quad - \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

[1]

Therefore,

$$x^3 + 6x^2 - 32 = (x - 2)(x^2 + 8x + 16)$$

correct quadratic factor [1]

Factorise $x^2 + 8x + 16$.

$$(x + 4)(x + 4)$$

Therefore,

$$x^3 + 6x^2 - 32 = (x-2)(x+4)^2$$

The roots to the equation $x^3 + 6x^2 - 32 = 0$ are:

$x = 2, \quad x = -4, \quad x = -4$ [1]

(4 marks)

(c) Give a reason why only one root is a valid solution of the logarithmic equations. Find the value of y corresponding to this root.

Answer

When $x = -4$, $\log_2 x$ and $\log_2 (x+2)$ will be:

$$\log_2 (-4) \text{ and } \log_2 (-4+2)$$

We cannot take the logarithm of a negative number.

There is no real value for $\log_2 (-4)$ and $\log_2 (-2)$ [1]

Therefore, the only root is $x = 2$.

Substitute $x = 2$ into $\log_2(y+1) = 3 - 2\log_2 x$

$$\log_2(y+1) = 3 - 2\log_2(2)$$

$$\log_2(y+1) = 1$$

Since $\log_2 2 = 1$

$$y + 1 = 2$$

$y = 1$ [1]

(2 marks)