IGCSE · Cambridge (CIE) · Further Maths

1 hour · 16 questions

Exam Questions

Quadratic Functions

Solving Quadratics by Factorising / Quadratic Formula / Completing the Square / Quadratic Equation Methods / Discriminants / Quadratic Graphs / Quadratic Inequalities

Medium (5 questions)	/20
Hard (6 questions)	/24
Very Hard (5 questions)	/27
Total Marks	**/71**

Medium Questions

1 (a) Write the expression $x^2 - 6x + 1$ in the form $(x + a)^2 + b$, where a and b are constants.

Answer

To complete the square, $x^2 + bx + c = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right) + c$

$$x^2 - 6x + 1 = (x-3)^2 - (3)^2 + 1$$

correct squared term $(x-3)^2$ [1]

Simplify.

$$= (x-3)^2 - 9 + 1$$

$\mathbf{(x-3)^2 - 8}$ [1]

(2 marks)

(b) Hence write down the coordinates of the minimum point on the curve $y = x^2 - 6x + 1$.

Answer

The minimum point on the curve $y = (x + a)^2 + b$ is $(-a, b)$.

For $y = (x-3)^2 - 8$ is

$\mathbf{(3, -8)}$ [1]

(1 mark)

2 Solve the inequality $(x-8)(x-10) > 35$.

Answer

Expand the double bracket

$$x^2 - 10x - 8x + 80 > 35$$

Collect the like terms on the left hand side of the inequality

$$x^2 - 18x + 80 > 35$$

Subtract 35 from both sides to make the right hand side 0

$$x^2 - 18x + 45 > 0$$

[1]

We want to find where the quadratic graph is greater than 0

Consider $y = x^2 - 18x + 45$ and draw a sketch

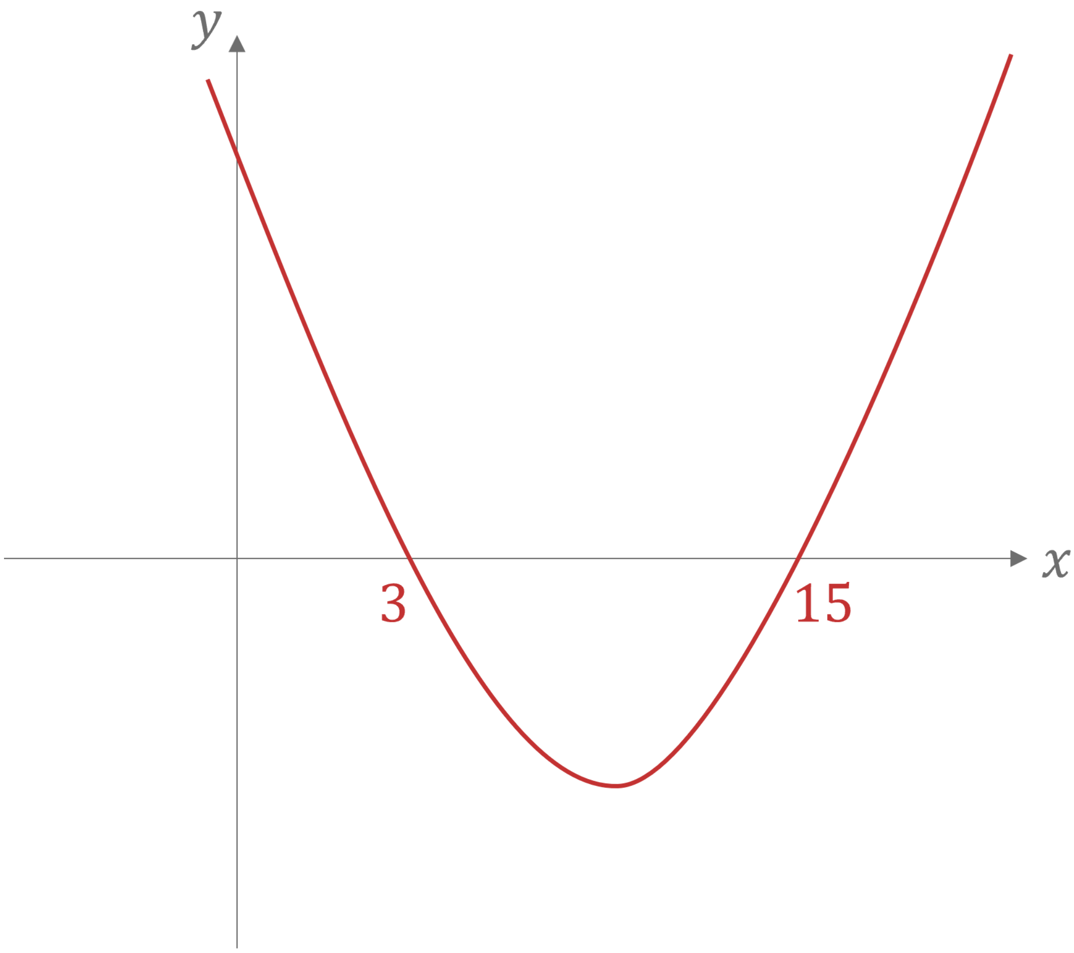
Identify the roots by factorising (or using the quadratic formula)

$$(x - 15)(x - 3) = 0$$

[1]

$$x = 15 \text{ and } x = 3$$

[1]



The quadratic is greater than 0 when the graph is above the x-axis

$x < 3$ or $x > 15$ [1]

$(-\infty, 3) \cup (15, \infty)$ is also accepted

(4 marks)

3 Solve the equation $2x - 11\sqrt{x} + 12 = 0$.

Answer

Rewrite the equation, using $\sqrt{x} = x^{\frac{1}{2}}$

$$2x - 11x^{\frac{1}{2}} + 12 = 0$$

Let $u = x^{\frac{1}{2}}$

$$2u^2 - 11u + 12 = 0$$

Factorise and solve.

$$(2u - 3)(u - 4) = 0$$

[1]

$$u = \frac{3}{2} \text{ and } u = 4$$

Replace u with $x^{\frac{1}{2}}$.

$$x^{\frac{1}{2}} = \frac{3}{2} \quad \text{and} \quad x^{\frac{1}{2}} = 4$$

[1]

Solve.

$$x = \left(\frac{3}{2}\right)^2$$

$$x = (4)^2$$

$x = \frac{9}{4}$ **and** $x = 16$ [1]

(3 marks)

4 Find the values of k for which the equation $x^2 + (k + 9)x + 9 = 0$ has two distinct real roots.

Answer

When an equation has two real roots, the discriminant is positive.

$$b^2 - 4ac > 0$$

$$(k+9)^2 - (4)(1)(9) > 0$$

[1]

Expand and simplify.

$$k^2 + 18x + 81 - 36 > 0$$
$$k^2 + 18x + 45 > 0$$

[1]

Factorise and solve $k^2 + 18x + 45 = 0$ to find critical values.

$$(k+15)(k+3) = 0$$

$$k = -15, \quad k = -3$$

[1]

Since we want where the positive quadratic is greater than 0, we want k values to the left of –15, and to the right of –3.

$\mathbf{k < -15 \quad or \quad k > -3}$ [1]

(4 marks)

5 (a) Find the values of the constant k for which the equation

$$2kx^2 + (k-6)x - 4 = 0$$

has no real solutions.

Answer

No real solutions means the discriminant is negative, $b^2 - 4ac < 0$

$$(k-6)^2 - 4(2k)(-4) < 0$$

[M1]

Expand and simplify, remember $(k-6)^2 = (k-6)(k-6)$

$$k^2 - 12k + 36 + 32k < 0$$
$$k^2 + 20k + 36 < 0$$

[A1]

Solve this quadratic inequality

First finding the critical points, e.g. by factorising

$$k^2 + 20k + 36 = 0$$
$$(k+18)(k+2) = 0$$

[M1]

$k = -18$ or $k = -2$

[A1]

$y = k^2 + 20k + 36$ is a positive quadratic (U-shape) so state the k values for where it is below x-axis ($y < 0$)

$-18 < k < -2$

[A1]
(5 marks)

(b) Hence determine the number of points of intersection, if any, between the line $y = 6x$ and the curve $y = 2kx^2 + kx - 4$ for $k > 0$.

Answer

The 'hence' means you need to relate this question to part (a)

Find the x-coordinates of the points of intersection (e.g. by setting the y's equal to each other)

$$2kx^2 + kx - 4 = 6x$$

Rearrange to the quadratic in part (a)

$$2kx^2 + (k-6)x - 4 = 0$$

You know this quadratic has no real solutions for $-18 < k < -2$ (the line and curve never intersect)

At the endpoints, $k = -18$ or $k = -2$, the line will be a tangent to the curve

However $k > 0$ is outside this range completely, so they must intersect twice

The straight line $y = 6x$ intersects the curve $y = 2kx^2 + kx - 4$ twice when $k > 0$

[B1]

(1 mark)

Hard Questions

1 Find the values of the constant k for which the equation $(2k - 1)x^2 + 6x + k + 1 = 0$ has real roots.

Answer

A quadratic equation has real roots if $b^2 - 4ac \geq 0$ $a = 2k - 1$, $b = 6$ and $c = k + 1$

$$6^2 - 4(2k - 1)(k + 1) \geq 0$$

[1]

Solve for *k*

$$36 - 4(2k^2 + k - 1) \geq 0$$
$$36 - 8k^2 - 4k + 4 \geq 0$$
$$8k^2 + 4k - 40 \leq 0$$

[1]

Factorise or use the quadratic formula to solve and find the critical values

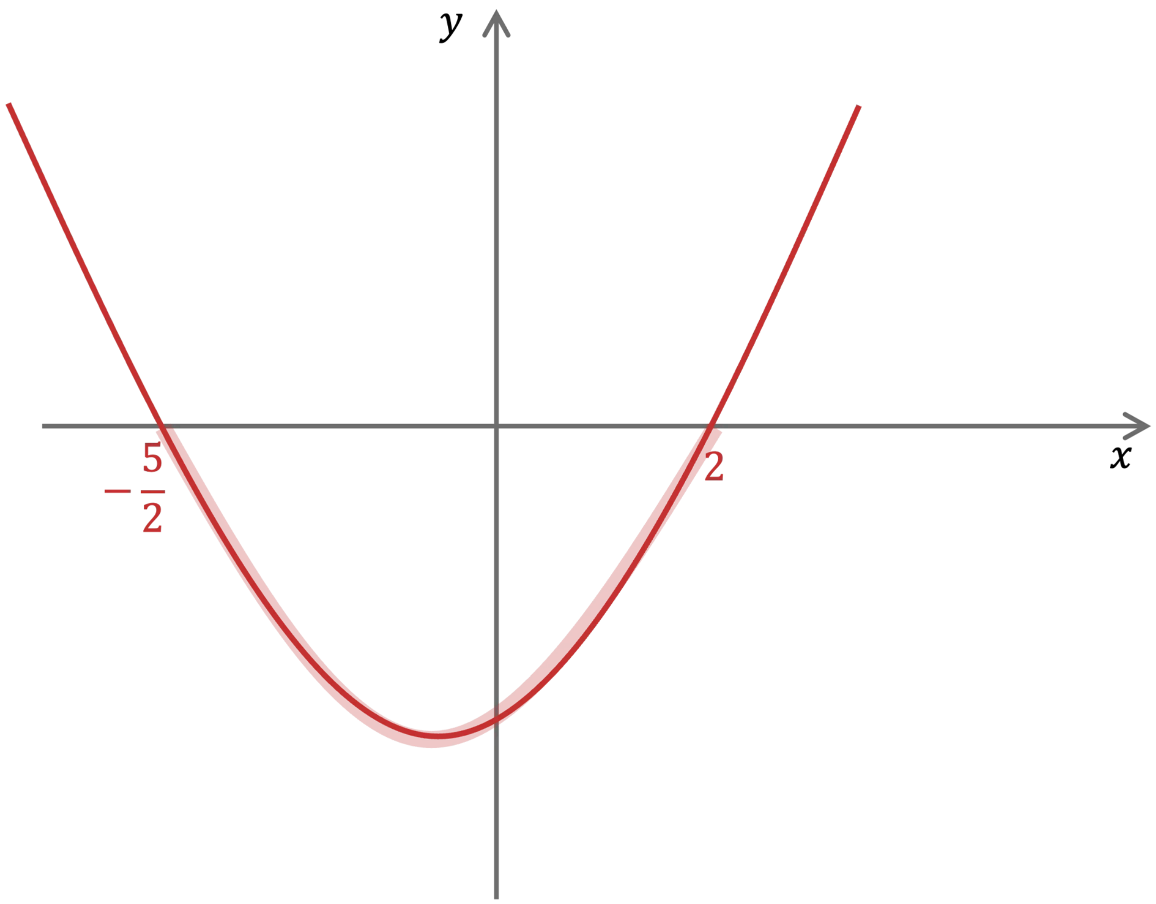
$$2k^2 + k - 10 \leq 0$$
$$\text{let } 2k^2 + k - 10 = 0$$
$$(k - 2)(2k + 5) = 0$$

[1]

$$k = 2,\ k = -\frac{5}{2}$$

both critical values found [1]

To decide which way the inequalities go, draw a very quick sketch of the graph of $8k^2 + 4k - 40 \leq 0$ with the critical values marked; it's a positive quadratic and we're looking for the values below the x-axis (or sketch $-8k^2 - 4k + 40 \geq 0$ as a negative parabola, and look for the region above the x-axis)



$-\frac{5}{2} \leq k \leq 2$ [1]

(5 marks)

2 Find the values of x for which $12x^2 - 20x + 5 < (2x + 1)(x - 1)$.

Answer

Expand the right-hand side of the inequality

$$12x^2 - 20x + 5 < (2x + 1)(x - 1)$$

$$12x^2 - 20x + 5 < 2x^2 - 2x + x - 1$$

Simplify

$$12x^2 - 20x + 5 < 2x^2 - x - 1$$

[1]

Make the quadratic inequality equal to 0 on one side

$$10x^2 - 19x + 6 < 0$$

Find the roots of the quadratic equation using the quadratic formula

$$x = \frac{2}{5} \text{ or } x = \frac{3}{2}$$

factorise or solve quadratic equation [1]

for correct values [1]

We have a positive quadratic and we are trying to see where the quadratic is strictly less than 0 (i.e. below the x-axis)

$$\frac{2}{5} < x < \frac{3}{2}$$ [1]

(4 marks)

3 (a) Write $9x^2 - 12x + 5$ in the form $p(x - q)^2 + r$, where p, q and r are constants.

Answer

Dividing the first two terms by the coefficient of x^2

$$9\left(x^2 - \frac{4}{3}x\right) + 5$$

Completing the square inside the bracket

$$9\left[\left(x - \frac{2}{3}\right)^2 - \frac{4}{9}\right] + 5$$

Expanding the square bracket gives

$$9\left(x - \frac{2}{3}\right)^2 - 4 + 5$$

$$9\left(x-\frac{2}{3}\right)^2+1$$

1 mark for each correct of p, q, and r [3]

(3 marks)

(b) Hence write down the coordinates of the minimum point of the curve $y = 9x^2 - 12x + 5$.

Answer

If $y = p(x - q)^2 + r$ then the turning point is at (q, r)

$$\left(\frac{2}{3}, 1\right)$$ [1]

(1 mark)

4 Solve $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 10 = 0$.

Answer

Let $y = x^{\frac{1}{3}}$, then

$$2y^2 - y - 10 = 0$$

$$(2y - 5)(y + 2) = 0$$

[1]

Solving the quadratic gives

$$y = \frac{5}{2} \text{ and } y = -2$$

Substituting $y = x^{\frac{1}{3}}$

$$x^{\frac{1}{3}} = \frac{5}{2} \text{ and } x^{\frac{1}{3}} = -2$$

[1]

Cubing both sides of each equation gives

$$x = \frac{125}{8}, \quad x = -8$$ [1]

(3 marks)

5 Find the set of values of k for which $4x^2 - 4kx + 2k + 3 = 0$ has no real roots.

Answer

A quadratic has no real roots when the discriminant is negative, since we cannot take the square root of a negative number.

Substitute the values from the equation into $b^2 - 4ac < 0$

$$(-4k)^2 - (4)(4)(2k+3) < 0$$

[1]

Expand and simplify.

$$16k^2 - 32k - 48 < 0$$

[1]

Divide through by 16.

$$k^2 - 2k - 3 < 0$$

Factorise and solve $k^2 - 2k - 3 = 0$ to find critical values.

$$(k+1)(k-3) = 0$$

[1]

$$k = -1 \text{ and } k = 3$$

[1]

We are solving a quadratic inequality, therefore we need to examine the graph of $y = k^2 - 2k - 3$ to determine which inequality signs are needed. Since we want the equation to be smaller than 0, we want the part of the graph **under** the x axis, between the two critical values.

Therefore,

$-1 < k < 3$ [1]

(5 marks)

6 Solve $6x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 1 = 0$.

Answer

This is a hidden quadratic equation which is easier to solve if we let $y = x^{\frac{1}{3}}$. This means that the quadratic becomes

$$6y^2 - 5y + 1 = 0$$

To factorise this quadratic, we know that the final terms in the brackets must both be −1, as they must have a product of +1. The first terms in the brackets must have a product of 6, so this must be 3 and 2, or 1 and 6. Upon inspection we see that the brackets must be

$$(3y - 1)(2y - 1) = 0$$

[1]

Solving these by setting each bracket equal to 0 gives

$$3y - 1 = 0 \qquad 2y - 1 = 0$$

$$y = \frac{1}{3} \qquad y = \frac{1}{2}$$

Using our previous substitution we have

$$x^{\frac{1}{3}} = \frac{1}{3} \qquad x^{\frac{1}{3}} = \frac{1}{2}$$

[1]

Cubing both sides of each equation gives us the final solutions

$$x = \frac{1}{27}, \; x = \frac{1}{8}$$ [1]

(3 marks)

Very Hard Questions

1 Find the values of k for which the line $y = kx + 3$ is a tangent to the curve $y = 2x^2 + 4x + k - 1$.

Answer

If the line meets the curve, they will have the same x and y-coordinate at one point. Set the curve equal to the line.

$$2x^2 + 4x + k - 1 = kx + 3$$

[1]

Bring all the terms to one side to equal the equation to 0:

$$2x^2 + 4x - kx + k - 1 - 3 = 0$$

[1]

Simplify:

$$2x^2 + (4 - k)x + k - 4 = 0$$

If the line is tangent to the curve, they meet once, therefore this quadratic equation will have 1 solution only. Use that the discriminant $b^2 - 4ac$ is equal to 0.

$$b^2 - 4ac = (4 - k)^2 - 4(2)(k - 4) = 0$$

[1]

$$16 - 8k + k^2 - 8k + 32 = 0$$

$$k^2 - 16k + 48 = 0$$

Solve the quadratic equation using the quadratic formula:

$$k = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(48)}}{2(1)}$$

$k = 12,\ k = 4$ [2]

(5 marks)

2 Find the values of k for which the line $y = x - 3$ intersects the curve $y = k^2x^2 + 5kx + 1$ at two distinct points.

Answer

Since they intersect, set the line equal to the curve

$$x - 3 = k^2x^2 + 5kx + 1$$

[1]

Rearrange the equation to make it equal to 0

Subtract x from both sides

$$-3 = k^2x^2 + (5k - 1)x + 1$$

Add 3 to both sides

$$0 = k^2x^2 + (5k - 1)x + 4$$

[1]

The line meets the curve twice (at 2 distinct points) so the discriminant, $b^2 - 4ac > 0$

Work out the discriminant

$$(5k - 1)^2 - 4(k^2)(4) > 0$$

[1]

$$(5k - 1)^2 - 16k^2 > 0$$

Expand the double bracket

$$25k^2 - 10k + 1 - 16k^2 > 0$$

Simplify

$$9k^2 - 10k + 1 > 0$$

[1]

Solve the quadratic inequality

Find the critical values using the quadratic formula (or by factorisation)

$$k = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(9)(1)}}{2(9)}$$

$$k = 1 \text{ or } \frac{1}{9}$$

[1]

We are trying to find where the (positive) quadratic is greater than zero, so consider where the graph is above the x-axis

$\mathbf{k < \frac{1}{9}}$ **or** $\mathbf{k > 1}$ [1]

(6 marks)

3 Find the exact values of the constant k for which the line $y = 2x + 1$ is a tangent to the curve $y = 4x^2 + kx + k - 2$.

Answer

Method 1

Put the equations equal to each other and simplify.

$$4x^2 + kx + k - 2 = 2x + 1$$

[1]

$$4x^2 + (k-2)x + (k-3) = 0$$

[1]

$y = 2x + 1$ and $y = 4x^2 + kx + k - 2$ only intersect once (because $y = 2x + 1$ is a tangent to the curve) and so $4x^2 + (k-2)x + (k-3) = 0$ must have only one real root and therefore $b^2 - 4ac = 0$

$$(k-2)^2 - (4)(4)(k-3) = 0$$

[1]

$$k^2 - 20k + 52 = 0$$

[1]

Solve using the quadratic formula.

$$k = \frac{-(-20) + \sqrt{(-20)^2 - (4)(1)(52)}}{2(1)}$$

[1]

$k = 10 + 4\sqrt{3}$ and $k = 10 - 4\sqrt{3}$ [1]

<u>Method 2</u>

Put the equations equal to each other.

$$4x^2 + kx + k - 2 = 2x + 1$$

The gradients of the curve and the line are equal because the line is a tangent to the curve, so differentiate both sides.

$$8x + k = 2$$

[1]

Rearrange to get an equation for k.

$$k = 2 - 8x$$

Substitute this into the original equation, simplify and rearrange.

$$4x^2 + (2-8x)x + (2-8x) - 2 = 2x + 1$$

[1]

$$4x^2 + 8x + 1 = 0$$

[1]

Solve using the quadratic formula.

$$x = \frac{-(8) + \sqrt{(8)^2 - (4)(4)(1)}}{2(4)}$$

[1]

$$x = -1 \pm \frac{\sqrt{48}}{8}$$

[1]

Substitute into $8x + k = 2$ and solve to find k.

$$8\left(-1 \pm \frac{\sqrt{48}}{8}\right) + k = 2$$

$k = 10 + 4\sqrt{3}$ and $k = 10 - 4\sqrt{3}$ [1]

(6 marks)

4 Find the values of k for which the line $y = kx - 7$ and the curve $y = 3x^2 + 8x + 5$ do not intersect.

Answer

Setting the two equations equal to each other

$$3x^2 + 8x + 5 = kx - 7$$

[1]

Collecting all terms to the left to set the quadratic equal to 0

$$3x^2 + 8x - kx + 12 = 0$$

$$3x^2 + (8 - k)x + 12 = 0$$

[1]

If the line and the curve do not intersect, then there are no solutions so the discriminant must be less than 0.

$$b^2 - 4ac < 0$$

$$a = 3,\ b = (8 - k),\ c = 12$$

$$(8 - k)^2 - 4(3)(12) < 0$$

[1]

Expanding the brackets and simplifying gives

$$64 - 16k + k^2 - 144 < 0$$

$$k^2 - 16k - 80 < 0$$

[1]

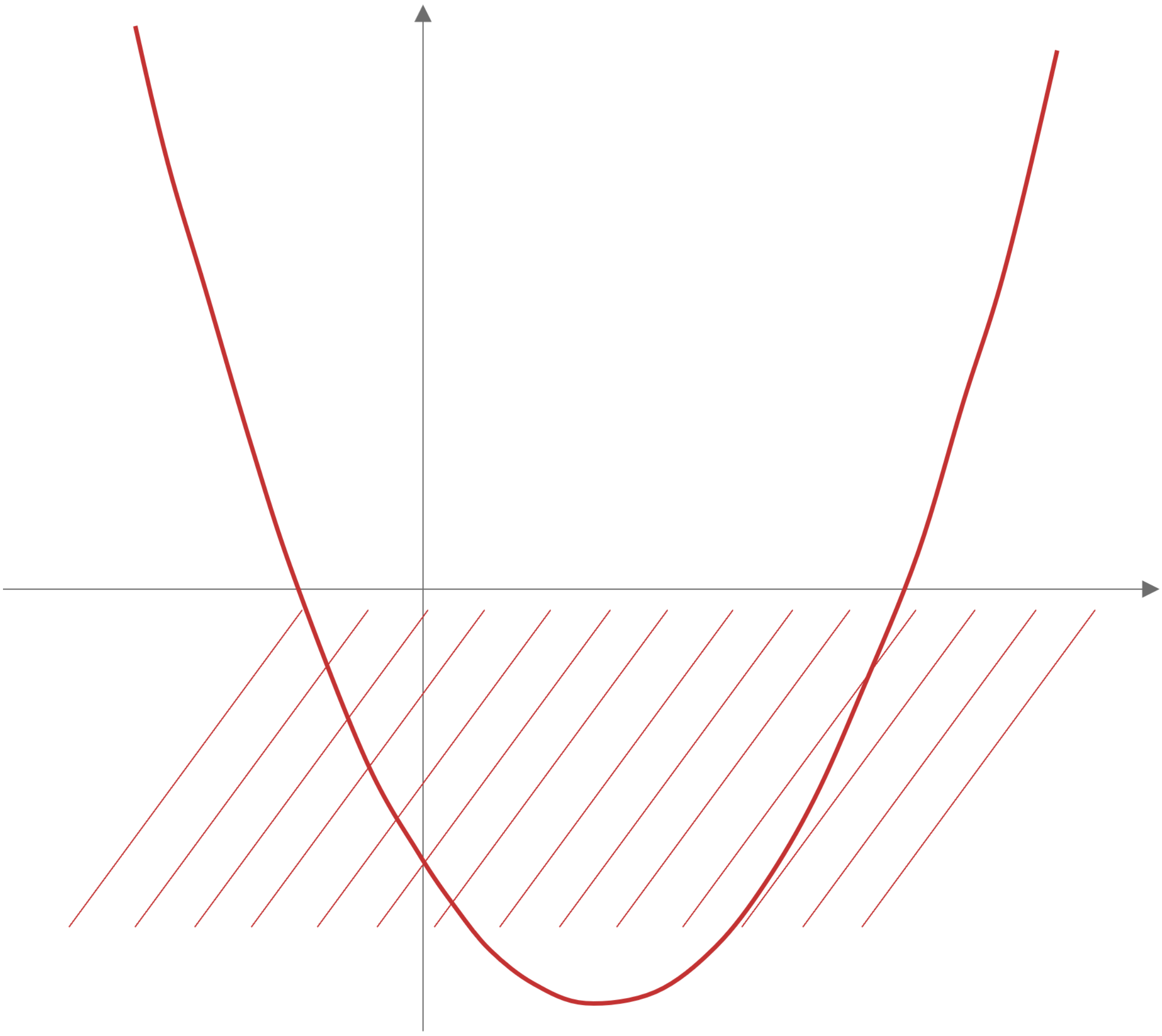
Factorising to find critical values

$$(k + 4)(k - 20) < 0$$

Critical values are $k = -4,\ k = 20$

[1]

To determine the correct region that satisfies the inequality, sketch a graph of the quadratic.



The region that satisfies the inequality is less than 0 so the correct region is below the x axis.

$-4 < k < 20$ [1]

(6 marks)

5 The curve $y = 2x^2 + k + 4$ intersects the straight line $y = (k+4)x$ at two distinct points. Find the possible values of k.

Answer

Substitute $y=(k+4)x$ into $y=2x^2+k+4$

$$(k+4)x = 2x^2 + k + 4$$

Rearrange.

$$2x^2 - (k+4)x + k + 4 = 0$$

[1]

Since the lines intersect at two distinct points, this means the discriminant of this equation must be greater than 0.

Substitute values from the equation into $b^2 - 4ac > 0$

$$(-(k+4))^2 - (4)(2)(k+4) > 0$$

[1]

Simplify.

$$k^2 + 8k + 16 - 8k - 32 > 0$$
$$k^2 - 16 > 0$$

Factorise and solve to find critical values.

$$(k+4)(k-4) = 0$$

$k = -4$ and $k = 4$

[1]

Since we want where the graph is greater than 0, we want values to the left of the negative root, and to the right of the positive root.

$k < -4$ and $k > 4$ [1]

(4 marks)