IGCSE · Cambridge (CIE) · Further Maths

2 hours · 11 questions

Exam Questions

Arithmetic & Geometric Progressions

Language of Sequences & Series / Arithmetic Progressions / Geometric Progressions

Medium (2 questions)	/17
Hard (7 questions)	/73
Very Hard (2 questions)	/22
Total Marks	**/112**

Medium Questions

1 (a) The 7th and 10th terms of an arithmetic progression are 158 and 149 respectively.

Find the common difference and the first term of the progression.

Answer

Using $u_n = a + (n-1)d$ write equations for the 7th and 10th terms of the arithmetic progression.

$$a + 6d = 158$$
$$a + 9d = 149$$

[1]

Solve the equations simultaneously.

$$\begin{array}{rr} & a + 9d = 149 \\ - & a + 6d = 158 \\ \hline & 3d = \ \ -9 \end{array}$$

$d = -3$ [1]

Substitute into one of the original equations.

$$a + (6)(-3) = 158$$

$a = 176$ [1]

(3 marks)

(b) Find the least number of terms of the progression for their sum to be negative.

Answer

Substitute the values found in part a into the formula to sum an arithmetic series.

$$S_n = \frac{n}{2}[2(176) + (n-1)(-3)]$$

$$S_n = \frac{n}{2}[352 - 3n + 3]$$

$$S_n = \frac{n}{2}[355 - 3n]$$

[1]

The sum of the progression needs to be negative.

$$\frac{n}{2}[355 - 3n] < 0$$

Solve – be careful, it's a quadratic inequality, so use a calculator or sketch the graph.

$n < 0$ and $n > 118.333...$

[1]

The value of n cannot be negative, so $n > 118.333...$ n has to be an integer

$n = 119$ [1]

(3 marks)

2 (a) The sum of the first 4 terms of an arithmetic progression is 38 and the sum of the next 4 terms is 86. Find the first term and the common difference.

Answer

The sum of an arithmetic progression is $S_n = \frac{n}{2}[2a + (n-1)d]$. Write an expression for the sum of the first four terms.

$$S_4 = \frac{4}{2}[2a + (4-1)d]$$

[1]

$$2[2a + 3d] = 38$$
$$2a + 3d = 19$$

[1]

Similarly, write an expression for the sum of the first eight terms.

$$S_8 = \frac{8}{2}[2a + (8-1)d]$$

$$S_8 = 4[2a + 7d]$$

We are told that the first four terms sum to 38, and the next four sum to 86. Therefore, the sum of the first 8 terms is the sum of 38 and 86.

$$4[2a + 7d] = 38 + 86$$

[1]

Expand and simplify.

$$2a + 7d = 31$$

Solve the equations simultaneously.

$$\begin{aligned} 2a+7d&=31 \\ -\quad 2a+3d&=19 \\ \hline 4d&=12 \\ d&=3 \end{aligned}$$

[1]

Substitute $d=3$ into one of the original equations and solve for a.

$$\begin{aligned} 2a+7(3)&=31 \\ 2a&=31-21 \\ a&=5 \end{aligned}$$

$a=5$ and $d=3$ [1]

(5 marks)

(b) The third term of a geometric progression is 12 and the sixth term is -96. Find the sum of the first 10 terms of this progression.

Answer

Using the nth term formula for a geometric progression, $u_n = ar^{n-1}$, the second term is

$$ar^2 = 12$$

[1]

The sixth term is

$$ar^5 = -96$$

[1]

Solve the equations simultaneously – rearrange the first and substitute into the second.

$$a = \frac{12}{r^2}$$

$$\frac{12}{r^2}(r^5) = -96$$

Solve for r.

$$12r^3 = -96$$
$$r^3 = -8$$
$$r = -2$$

[1]

Substitute into either of the original equations to find a.

$$a(-2)^2 = 12$$
$$a = 3$$

[1]

The sum of a geometric progression is $S_n = \dfrac{a(1-r^n)}{1-r}$.

$$S_{10} = \frac{3(1-(-2)^{10})}{1-(-2)}$$

[1]

$S_{10} = -1023$ [1]

(6 marks)

Hard Questions

1 (a) In an arithmetic progression, the 5th term is equal to $\frac{1}{3}$ of the 16th term. The sum of the 5th term and the 16th term is equal to 33.

Find the sum of the first 10 terms of this progression.

Answer

Using the general equation for the *n*th term, $u_n = a + (n-1)d$ where a is the first term and d is the common difference, we will form two simultaneous equations in a and d.
We can write the 5th term as

$$u_5 = a + (5-1)d$$
$$u_5 = a + 4d$$

and the 16th term as

$$u_{16} = a + (16-1)d$$
$$u_{16} = a + 15d$$

The question states "the 5th term is equal to $\frac{1}{3}$ of the 16th term" so

$$a + 4d = \frac{1}{3}(a + 15d)$$

[1]

Simplify.

$$a + 4d = \frac{1}{3}a + 5d$$
$$\frac{2}{3}a = d$$

Use the information that "the sum of the 5th term and the 16th term is equal to 33".

$$(a+4d)+(a+15d)=33$$

[1]

Simplify.

$$2a+19d=33$$

Substitute $d=\frac{2}{3}a$ into this equation.

$$2a+19\left(\frac{2}{3}a\right)=33$$

Solve for a.

$$\frac{44}{3}a=33$$

$$a=\frac{99}{44}$$

Simplify.

$$a=\frac{9}{4}$$

[1]

Substitute $a=\frac{9}{4}$ into $\frac{2}{3}a=d$.

$$\frac{2}{3}\times\frac{9}{4}=d$$

$$d=\frac{18}{12}$$

Simplify.

$$d=\frac{3}{2}$$

[1]

We haven't finished! The question asks for the sum of the first 10 terms.

Use the formula $S_n = \frac{n}{2}(2a + (n-1)d)$.

$$S_{10} = \frac{10}{2}\left(2 \times \frac{9}{4} + 9 \times \frac{3}{2}\right)$$

[1]

Evaluate.

$$S_{10} = 5\left(\frac{18}{4} + \frac{27}{2}\right)$$
$$S_{10} = 5 \times 18$$

$S_{10} = 90$ [1]

(6 marks)

(b) In a geometric progression, the sum of the first two terms is equal to 16. The sum to infinity is equal to 25.

Find the possible values of the first term.

Answer

If a is the first term and r is the common difference then the sum of the first two terms being 16 can be expressed as

$$a + ar = 16$$

[1]

Using $S_\infty = \frac{a}{1-r}$, the sum to infinity is 25.

$$\frac{a}{1-r} = 25$$

[1]

We can isolate a in both the above equations. Starting with $a + ar = 16$, factorise the left hand side.

$$a(1 + r) = 16$$

Divide by $1 + r$.

$$a = \frac{16}{1 + r}$$

Turning our attention to $\frac{a}{1 - r} = 25$, multiply by the denominator.

$$a = 25(1 - r)$$

Now equate both the above.

$$\frac{16}{1 + r} = 25(1 - r)$$

Multiply by $1 + r$.

$$16 = 25(1 - r)(1 + r)$$

[1]

Solve.

$$\frac{16}{25} = (1 - r)(1 + r)$$
$$\frac{16}{25} = 1 - r^2$$
$$r^2 = \frac{9}{25}$$
$$r = \pm\frac{3}{5}$$

[1]

Now find the possible values of a by substituting both versions of r into $a = \dfrac{16}{1+r}$.

$$r = \frac{3}{5},\ a = \frac{16}{1+\frac{3}{5}} = 16 \div \frac{8}{5} = 16 \times \frac{5}{8} = 2 \times 5$$

a = 10 [1]

$$r = -\frac{3}{5},\ a = \frac{16}{1-\frac{3}{5}} = 16 \div \frac{2}{5} = 16 \times \frac{5}{2} = 8 \times 5$$

a = 40 [1]

(6 marks)

2 (a) The sum of the first two terms of a geometric progression is 10 and the third term is 9.

(i) Find the possible values of the common ratio and the first term.

[5]

(ii) Find the sum to infinity of the convergent progression.

[1]

Answer

i) The first term is a and the second term is ar. The sum of the first 2 terms is 10.

$$a + ar = 10$$

[1]

The third term is ar^2. The third term is equal to 9.

$$ar^2 = 9$$

[1]

Rearrange to make a the subject.

$$a = \frac{9}{r^2}$$

Solve the equations simultaneously.

$$\frac{9}{r^2} + \left(\frac{9}{r^2}\right)r = 10$$

Multiply through by r^2.

$$9 + 9r = 10r^2$$

Rearrange.

$$10r^2 - 9r - 9 = 0$$

Solve the quadratic equation.

$$r = \frac{3}{2}, -\frac{3}{5}$$

[1]

Substitute r in one of the original equations to find a at each point.

$$a = \frac{9}{\left(\frac{3}{2}\right)^2} = 4 \quad a = \frac{9}{\left(-\frac{3}{5}\right)^2} = 25$$

$\boldsymbol{r = -\frac{3}{5}, \frac{3}{2}}$ and $\boldsymbol{a = 25, 4}$ [1]

ii) Use the sum to infinity formula for a geometric series, $S_\infty = \frac{a}{1-r}$ $(|r| < 1)$ with $r = -\frac{3}{5}$ and $a = 25$ because this is the only case where the modulus of r is less than 1.

$$S_\infty = \frac{25}{1-\left(-\frac{3}{5}\right)} = \frac{125}{8}$$

$\boldsymbol{\frac{125}{8}}$ [1]

(6 marks)

(b) In an arithmetic progression, $u_1 = -10$ and $u_4 = 14$. Find $u_{100} + u_{101} + u_{102} + \ldots + u_{200}$ the sum of the 100th to the 200th terms of the progression.

Answer

This is an arithmetic progression.
Work out the common difference.

$$d = \frac{14-(-10)}{3} = 8$$

[1]

The sum of the 100th to 200th terms will be equal to the sum of the first 200 terms subtract the sum of the first 99 terms.

$$S_{200} - S_{99} = \frac{1}{2}(200)(2a + (n-1)d) - \frac{1}{2}(99)(2a + (n-1)d)$$

$$S_{200} - S_{99} = \frac{1}{2}(200)(2(-10) + (200-1)8) - \frac{1}{2}(99)(2(-10) + (99-1)8)$$

[2]

$$S_{200} - S_{99} = (100 \times 1572) - (49.5 \times 764)$$

$$S_{200} - S_{99} = 119382$$

119382 [1]

(4 marks)

3 (a) An arithmetic progression has a second term of -14 and a sum to 21 terms of 84. Find the first term and the 21st term of this progression.

Answer

Using the formula $a + (n-1)d$ the second term is

$$a + d = -14$$

[1]

Using the formula for the sum of an arithmetic progression $S_n = \frac{n}{2}(2a + (n-1)d)$ we have

$$84 = \frac{21}{2}(2a + 20d)$$

Simplify.

$$8 = 2a + 20d$$

$$4 = a + 10d$$

[1]

Solve the equations simultaneously. (Subtract first equation from the second.)

$$9d = 18$$

$$d = 2$$

[1]

Substitute into either of the original equations.

$$a + 2 = -14$$

$$a = -16$$

[1]

We can now find the 21st term.

$$u_{21} = -16 + 20 \times 2$$

24 [1]

(5 marks)

(b) A geometric progression has a second term of $27p^2$ and a fifth term of p^5 . The common ratio, r, is such that $0 < r < 1$.

(i) Find r in terms of p.

[2]

(ii) Hence find, in terms of p, the sum to infinity of the progression.

[3]

(iii) Given that the sum to infinity is 81, find the value of p.

[2]

Answer

i) Using the formula $a_n = ar^{n-1}$ the second term is

$$ar = 27p^2$$

The fifth term is

$$ar^4 = p^5$$

both equations [1]

Divide the fifth term by the second term.

$$\frac{ar^4}{ar} = \frac{p^5}{27p^2}$$

Simplify.

$$r^3 = \frac{p^3}{27}$$

Take the cube root of both sides.

$$r = \frac{p}{3}$$ [1]

ii) Substitute the previous answer into the first equation to find a.

$$a \times \frac{p}{3} = 27p^2$$

[1]

$$a = 81p$$

[1]

Use the formula for the sum to infinity when $|r| < 1$, $S_\infty = \frac{a}{1 - r}$.

$$S_\infty = \frac{81p}{1 - \frac{p}{3}}$$

Multiply the numerator and denominator by 3.

$$\frac{243}{3 - p}$$ [1]

iii) Set the previous answer equal to 81.

$$81 = \frac{243}{3 - p}$$

[1]

Multiply both sides by $3 - p$.

$$81(3 - p) = 243$$

Expand the brackets.

$$243 - 81p = 243p$$

Collect like terms.

$$243 = 324p$$

$$p = \frac{3}{4}$$ [1]

(7 marks)

4 (a) The first 5 terms of a sequence are given below.

4 -2 1 -0.5 0.25

(i) Find the 20th term of the sequence.

[2]

(ii) Explain why the sum to infinity exists for this sequence and find the value of this sum.

[2]

Answer

i) To get from term to term, divide by -2 which is the same as multiplying by $-\frac{1}{2}$. So this is a geometric series with

$$a = 4, \quad r = -\frac{1}{2}$$

Use the formula $u_n = ar^{n-1}$ with $n = 20$ to work out the 20th term.

$$u_{20} = 4 \times \left(-\frac{1}{2}\right)^{20-1}$$

[1]

$$u_{20} = -\frac{1}{131072}$$ [1]

ii) The sum to infinity exists if the modulus of the common ratio is less than 1, i.e. $|r| < 1$

$$|r| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

$|r| < 1$ so the sum to infinity does exist [1]

Use the sum to infinity formula, $S_\infty = \frac{a}{1-r}$.

$$S_\infty = \frac{4}{1-\left(-\frac{1}{2}\right)} = \frac{8}{3}$$

$S_\infty = \frac{8}{3}$ [1]

(4 marks)

(b) The tenth term of an arithmetic progression is 15 times the second term. The sum of the first 6 terms of the progression is 87.

(i) Find the common difference of the progression.

[4]

(ii) For this progression, the nth term is 6990. Find the value of n.

[3]

Answer

i) Use the information in the question to make 2 equations.

For the first equation, use "the tenth term ... is 15 times the second term".

$$u_{10} = 15 \times u_2$$

Use $u_n = a + (n-1)d$ for the tenth and second terms and set equal to each other.

$$a + (10-1)d = 15 \times (a + (2-1)d)$$
$$a + 9d = 15(a + d)$$

[1]

$$a + 9d = 15a + 15d$$
$$14a = -6d$$
$$a = -\frac{3}{7}d$$

For the second equation use "the sum of the first 6 terms ... is 87". "$S_n = \frac{1}{2}n(2a + (n-1)d)$"

$$S_6 = \frac{1}{2} \times 6(2a + 5d) = 87$$

[1]

$$3(2a + 5d) = 87$$
$$2a + 5d = 29$$

Solve the two equations simultaneously.

$$2\left(-\frac{3}{7}d\right) + 5d = 29$$

[1]

$$-\frac{6}{7}d + 5d = 29$$
$$\frac{29}{7}d = 29$$

$d = 7$ [1]

ii) From (i), $d = 7$ and $a = -\frac{3}{7}d$. Find a.

$$a = -\frac{3}{7} \times 7 = -3$$

[1]

Use the formula $u_n = a + (n-1)d$ to set the nth term equal to 6990.

$$6990 = -3 + (n-1)7$$

[1]

$$6993 = 7n - 7$$
$$7000 = 7n$$

$n = 1000$ [1]

(7 marks)

5 (a) An arithmetic progression has a first term of 7 and a common difference of 0.4. Find the least number of terms so that the sum of the progression is greater than 300.

Answer

To sum an arithmetic series, $S_n = \frac{n}{2}[2a + (n-1)d]$.

Substitute the known values into the equation.

$$S_n = \frac{n}{2}[2(7) + (n-1)(0.4)]$$

$$S_n = \frac{n}{2}[14 + (n-1)(0.4)]$$

[1]

Expand and simplify.

$$S_n = \frac{n}{2}[13.6 + 0.4n]$$

$$S_n = \frac{13.6n + 0.4n^2}{2}$$

When the sum is greater than 300,

$$\frac{13.6n + 0.4n^2}{2} > 300$$

[1]

Rearrange.

$$13.6n + 0.4n^2 > 600$$

$$0.4n^2 + 13.6n - 600 > 0$$

Use the quadratic formula to find the positive critical value. We don't need the negative one because we can't have a negative number of terms.

$$n = \frac{-(13.6) \pm \sqrt{(13.6)^2 - (4)(0.4)(-600)}}{2(0.4)}$$

$$n = \frac{-(13.6) \pm \sqrt{1144.96}}{0.8}$$

$$n = 25.2966 \quad \text{and} \quad n = -59.2966$$

[1]

The term number, n, needs to be a positive integer so reject $n = -59.2966$ and round 25.2966 up (since $n > 25.2966$ and the least number of terms is required)

26 terms [1]

(4 marks)

(b) The sum of the first two terms of a geometric progression is 9 and its sum to infinity is 36. Given that the terms of the progression are positive, find the common ratio.

Answer

The sum of the first two terms of a geometric progression can be written as $a + ar$.

$$a + ar = 9$$

[1]

The sum to infinity is 36.

$$\frac{a}{1 - r} = 36$$

[1]

Rearrange to make a the subject.

$$a = 36(1 - r)$$

Substitute into the first equation.

$$36(1 - r) + 36(1 - r)r = 9$$

Expand and simplify.

$$36 - 36r + 36r - 36r^2 = 9$$
$$36 - 36r^2 = 9$$
$$36(1 - r^2) = 9$$

[1]

Solve to find r. We are told that the terms of the progression are positive, and so therefore r will be positive and we can ignore any negative solutions.

$$1 - r^2 = \frac{1}{4}$$
$$r^2 = \frac{3}{4}$$
$$r = \sqrt{\frac{3}{4}}$$

$\boldsymbol{r = \frac{\sqrt{3}}{2}}$ [1]

(4 marks)

6 (a) An arithmetic progression has a second term of 8 and a fourth term of 18. Find the least number of terms for which the sum of this progression is greater than 1560.

Answer

The n^{th} term in an arithmetic progression is $u_n = a + (n-1)d$.

$$u_2 = a + d = 8$$

$$u_4 = a + 3d = 18$$

[1]

Solve simultaneously. (Subtract first equation from second.)

$$2d = 10$$

$$d = 5$$

[1]

Substitute into either of the original equations.

$$a + 5 = 8$$

$$a = 3$$

[1]

The sum of an arithmetic progression is $S_n = \frac{n}{2}(2a + (n-1)d)$.

$$\frac{n}{2}(2 \times 3 + (n-1) \times 5) > 1560$$

Simplify.

$$\frac{n}{2}(6 + 5n - 5) > 1560$$

[1]

Simplify further and multiply by 2.

$$n(5n + 1) > 3120$$

Expand the bracket and subtract 1560 from both sides.

$$5n^2 + n - 3120 > 0$$

[1]

Solve the quadratic – use the formula and/or a calculator.

$$n > 24.9, \ n < -25.1$$

n needs to be a positive integer so round 24.9 up.

$n = 25$ [1]

(6 marks)

(b) A geometric progression has a sum to infinity of 72. The sum of the first 3 terms of this progression is $\frac{333}{8}$.

(i) Find the value of the common ratio.

[5]

(ii) Hence find the value of the first term.

[1]

Answer

i) Use the formula for the sum to infinity $\frac{a}{1 - r}$.

$$\frac{a}{1 - r} = 72$$

Multiply through by the denominator.

$$a = 72(1 - r)$$

Use the formula $S_n = \frac{a(1 - r^n)}{1 - r}$ for the rest of the information.

$$\frac{a(1 - r^3)}{1 - r} = \frac{333}{8}$$

[1]

Substitute the expression for a.

$$\frac{72(1 - r)(1 - r^3)}{1 - r} = \frac{333}{8}$$

[1]

Cancel $(1 - r)$ on the left-hand side.

$$72(1 - r^3) = \frac{333}{8}$$

Divide both sides by 72.

$$1 - r^3 = \frac{37}{64}$$

[1]

Rearrange.

$$\frac{27}{64} = r^3$$

[1]

Cube root both sides.

$r = \frac{3}{4}$ [1]

ii) Substitute our value of r from part (i) into the sum to infinity equation.

$$\frac{a}{1 - \frac{3}{4}} = 72$$

Simplify.

$$\frac{a}{\frac{1}{4}} = 72$$

Multiply both sides by $\frac{1}{4}$.

$a = 18$ [1]

(6 marks)

7 (a) A geometric progression has a first term of 3 and a second term of 2.4. For this progression, find the sum of the first 8 terms.

Answer

This is a geometric progression with first term.

$$a = 3$$

Work out the common ratio by dividing the second term by the first.

$$r = \frac{2.4}{3}$$

$$r = 0.8$$

[1]

Use the sum of a geometric progression formula, $S_n = \frac{a(1 - r^n)}{1 - r}$, to calculate the sum of the first 8 terms.

$$S_8 = \frac{3(1 - 0.8^8)}{1 - 0.8}$$

[1]

$S_8 = 12.48$ [1]

12.5 is accepted

(3 marks)

(b) Find the sum to infinity.

Answer

From part (a) we know that $a = 3$, $r = 0.8$.

Substitute these values into the sum to infinity formula, $S_\infty = \frac{a}{1 - r}$.

$$S_\infty = \frac{3}{1-0.8}$$

15 [1]
(1 mark)

(c) Find the least number of terms for which the sum is greater than 95% of the sum to infinity.

Answer

Calculate 95% of the sum to infinity using the result from part (b).

$$0.95 \times 15 = 14.25$$

Set up an inequality using the sum of the first n terms, $S_n = \frac{a(1-r^n)}{1-r}$.

$$\frac{a(1-r^n)}{1-r} > 14.25$$

Substitute the values $a=3$, $r=0.8$ from part (a).

$$\frac{3(1-0.8^n)}{1-0.8} > 14.25$$

[1]

Simplify the denominator.

$$\frac{3(1-0.8^n)}{0.2} > 14.25$$

Multiply both sides by 0.2.

$$3(1-0.8^n) > 2.85$$

Divide both sides by 3.

$$1-0.8^n > 0.95$$

Add 0.8^n to both sides and subtract 0.95.

$$0.05 > 0.8^n$$

[1]

Take logarithms of both sides.

$$\log_{10}(0.05) > \log_{10}(0.8^n)$$

Use the law of logarithms, $\log a^m = m \log a$, to simplify.

$$\log_{10}(0.05) > n\log_{10}(0.8)$$

Divide both sides by $\log_{10}(0.8)$. This is a negative value so reverse the inequality sign.

$$\frac{\log_{10}(0.05)}{\log_{10}(0.8)} < n$$

[1]

Calculate.

$$n > 13.425...$$

n is an integer so write down the smallest integer that is greater than 13.425...

$n = 14$ [1]

(4 marks)

Very Hard Questions

1 (a) The 2nd, 8th and 44th terms of an arithmetic progression form the first three terms of a geometric progression. In the arithmetic progression, the first term is 1 and the common difference is positive.

(i) Show that the common difference of the arithmetic progression is 5.

(ii) Find the sum of the first 20 terms of the arithmetic progression.

Answer

(i) Find expressions for the first three terms of the geometric sequence using the fact they are the 2nd, 8th and 44th terms of an arithmetic sequence ($u_n = a + (n-1)d$).

$$1 + d, \quad 1 + 7d, \quad 1 + 43d$$

[1]

A geometric series has a common ratio, so the second term divided by the first will be equal to the third term divided by the second.

$$\frac{1+7d}{1+d} = \frac{1+43d}{1+7d}$$

1 mark for each fraction [2]

Rearrange and simplify.

$$(1+7d)^2 = (1+43d)(1+d)$$
$$1 + 14d + 49d^2 = 1 + 44d + 43d^2$$
$$6d^2 - 30d = 0$$

[1]

Factorise and solve.

$$6d(d-5) = 0$$
$$d = 0 \text{ and } d = 5$$

The common difference cannot be 0.

$d = 5$ [1]

(ii) Substitute $a = 1$ and $d = 5$ into the sum of an arithmetic progression formula $S_n = \frac{n}{2}(2a + (n-1)d)$.

$$S_{20} = \frac{20}{2}(2(1) + (20-1)(5))$$

[1]

Calculate.

$S_{20} = 970$ [1]

(7 marks)

(b) (i) Find the 5th term of the geometric progression.

(ii) Explain whether or not the sum to infinity of this geometric progression exists.

Answer

(i) Using $d = 5$, we know the geometric progression starts $6, 36, 216$.

$$\therefore\ r = 6$$

Use $u_n = ar^{n-1}$ to find the 5th term.

$$u_{20} = (6)(6^{5-1})$$

7776 [1]

(ii) We can only sum a geometric progression to infinity when the common ratio is less than 1.

The sum to infinity does not exist for this geometric progression because the common ratio is greater than 1 ($r = 6$) [1]

(3 marks)

2 (a) In an arithmetic sequence, the 20th term is 10% of the sum of the first 25 terms.

The sum of the 20th term and the 21st term is 73.

Find the 8th term.

Answer

Substitute $n = 20$ into the formula $u_n = a + (n-1)d$

$$u_{20} = a + (20-1)d$$
$$= a + 19d$$

Substitute $n = 25$ into the formula $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{25} = \frac{25}{2}[2a + (25-1)d]$$
$$= \frac{25}{2}(2a + 24d)$$

Substitute the two expressions above into the 10% relationship from the question, $u_{20} = 0.1S_{25}$

$$a + 19d = 0.1 \times \frac{25}{2}(2a + 24d)$$

[M1]

Make a (or d) the subject

$$a + 19d = \frac{5}{4}(2a + 24d)$$
$$a + 19d = \frac{5}{2}(a + 12d)$$
$$2(a + 19d) = 5(a + 12d)$$
$$2a + 38d = 5a + 60d$$
$$-3a = 22d$$
$$a = -\frac{22}{3}d$$

[A1]

Find an expression for the sum of the 20th and the 21st term using $u_n = a + (n-1)d$

$$(a+19d)+(a+20d)$$

Set this equal to 73

$$(a+19d)+(a+20d)=73$$

[M1]

Simplify the equation above then substitute in $a = -\frac{22}{3}d$ to find d

$$2a+39d=73$$
$$2\left(-\frac{22}{3}d\right)+39d=73$$
$$\frac{73d}{3}=73$$
$$d=3$$

[A1]

Substitute this value of d into $a = -\frac{22}{3}d$ to find a

$$a=-22$$

[A1]

Substitute $a=-22$ and $d=3$ into $u_n = a+(n-1)d$ where $n=8$ to find the 8th term

$$-22+(8-1)\times 3$$

The 8th term is -1

[A1]

(6 marks)

(b) In a geometric sequence, the sum to infinity is 64 and the sum of the first 7 terms is $\frac{127}{2}$.

Find the ratio of the 4th term to the 7th term, giving your answer in the form $m : 1$.

Answer

Use the formula $S_\infty = \frac{a}{1-r}$ to form an equation in a and r (equation 1)

$$\frac{a}{1-r} = 64 \qquad \textcircled{1}$$

[B1]

Use the formula $S_n = \frac{a(1-r^n)}{1-r}$ where $n = 7$ and $S_7 = \frac{127}{2}$ to form another equation in a and r (equation 2)

$$\frac{a(1-r^7)}{1-r} = \frac{127}{2} \qquad \textcircled{2}$$

[B1]

Make a the subject of equation 1 and substitute it into equation 2

$$\frac{64(1-r)(1-r^7)}{1-r} = \frac{127}{2}$$

$$\frac{64\cancel{(1-r)}(1-r^7)}{\cancel{1-r}} = \frac{127}{2}$$

$$64(1-r^7) = \frac{127}{2}$$

[M1]

Solve for r

$$1 - r^7 = \frac{127}{128}$$

$$1 - \frac{127}{128} = r^7$$

$$\frac{1}{128} = r^7$$

$$\sqrt[7]{\frac{1}{128}} = r$$

$$r = \frac{1}{2}$$

[A1]

Use $u_n = ar^{n-1}$ to form a ratio between the 4th and 7th terms

$$ar^3 : ar^6$$

[M1]

Cancel the a and r^3 from both sides and substitute in $r = \frac{1}{2}$

$$1 : r^3$$

$$1 : \left(\frac{1}{2}\right)^3$$

$$1 : \frac{1}{8}$$

Write this in the form $m : 1$ by multiplying both sides by 8

$8 : 1$

[A1]

(6 marks)