IGCSE · Cambridge (CIE) · Further Maths 1 hour 12 questions

Exam Questions

Binomial Theorem

Binomial Theorem

Medium (3 questions)	/13
Hard (5 questions)	/27
Very Hard (4 questions)	/26
Total Marks	**/66**

Medium Questions

1 (a) Find the first 3 terms in the expansion of $\left(4-\frac{x}{16}\right)^6$ in ascending powers of x. Give each term in its simplest form.

Answer

Use the Binomial Theorem to expand the polynomial

$$\left(4-\frac{x}{16}\right)^6 = 4^6 + \binom{6}{1}4^5\left(-\frac{x}{16}\right) + \binom{6}{2}4^4\left(-\frac{x}{16}\right)^2 + \ldots$$

$$\left(4-\frac{x}{16}\right)^6 = 4096 - 384x + 15x^2 + \ldots$$

$4096 - 384x + 15x^2$ [3]

1 mark for each correct term

(3 marks)

(b) Hence find the term independent of x in the expansion of $\left(4-\frac{x}{16}\right)^6\left(x-\frac{1}{x}\right)^2$

Answer

From part (a) we know that $\left(4-\frac{x}{16}\right)^6 = 4096 - 384x + 15x^2 + \ldots$ so

$$\left(4-\frac{x}{16}\right)^6\left(x-\frac{1}{x}\right)^2 = (4096 - 384x + 15x^2 + \ldots)\left(x-\frac{1}{x}\right)^2$$

Expand the second bracket

$$\left(x-\frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$$

Therefore

$$\left(4-\frac{x}{16}\right)^6\left(x-\frac{1}{x}\right)^2 = (4096 - 384x + 15x^2 + \ldots)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

[1]

Look at the term without (independent of) an x

The term independent of x will come from 2 different places

$$4096 \times -2 = -8192$$

and

$$15x^2 \times \frac{1}{x^2} = 15$$

[1]

Add these together

$$-8192 + 15 = -8177$$

-8177 [1]

(3 marks)

2 Find the term independent of x in the binomial expansion of $\left(3x - \frac{1}{x}\right)^6$.

Answer

The only term in the expansion that will include no x is the x^3 one, since this is where the powers of x cancel each other out.

Find the x^3 term.

$$^6C_3 \times (3x)^3 \times \left(-\frac{1}{x}\right)^3$$

$$20 \times (3x)^3 \times \left(-\frac{1}{x}\right)^3$$

[1]

Simplify.

$$20 \times 27x^3 \times \left(-\frac{1}{x^3}\right)$$

$$\frac{-540x^3}{x^3}$$

-540 [1]

(2 marks)

3 Find the coefficient of x^2 in the expansion of $\left(x-\frac{3}{x}\right)\left(x+\frac{2}{x}\right)^5$.

Answer

Expand $\left(x+\frac{2}{x}\right)^5$

$$(1)(x)^5+({}^5C_1)(x)^4\left(\frac{2}{x}\right)^1+({}^5C_2)(x)^3\left(\frac{2}{x}\right)^2+\ldots$$

Simplify.

$$x^5+10x^3+40x+\ldots$$

for two correct terms [1]

for $10x^3$ [1]

for $40x$ [1]

When multiplied by $\left(x-\frac{3}{x}\right)$, the only terms in x^2 will be when $40x$ is multiplied by x and when $10x^3$ is multiplied by $-\frac{3}{x}$

Therefore, the coefficient of x^2 is

$$(1\times 40)-(3\times 10)$$

[1]

10 [1]

(5 marks)

Hard Questions

1 (a) Expand $(2-x)^5$, simplifying each coefficient.

Answer

Using the formula for the binomial theorem with $a = 2,\ b = -x,\ n = 5$

$$2^5 + {}^5C_1 2^4(-x) + {}^5C_2 2^3(-x)^2 + {}^5C_3 2^2(-x)^3 + {}^5C_4 2^1(-x)^4 + {}^5C_5(-x)^5$$

$$32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$ [3]

2 marks for any four or five terms correct, or 1 mark for any three terms correct

(3 marks)

(b) Hence solve $\dfrac{e^{(2-x)^5} \times e^{80x}}{e^{10x^4+32}} = e^{-x^5}$.

Answer

Multiplying both sides of the equation by the denominator

$$e^{(2-x)^5} \times e^{80x} = e^{-x^5}\left(e^{10x^4+32}\right)$$

Expanding the brackets and applying the laws of indices

$$e^{(2-x)^5 + 80x} = e^{-x^5 + 10x^4 + 32}$$

Taking logs of both sides

$$(2-x)^5 + 80x = -x^5 + 10x^4 + 32$$

[1]

$$(2-x)^5 = -x^5 + 10x^4 - 80x + 32$$

Comparing this to the expansion in part (a) gives

$$32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5 = -x^5 + 10x^4 - 80x + 32$$

$$\cancel{32} \cancel{- 80x} + 80x^2 - 40x^3 \cancel{+ 10x^4} \cancel{- x^5} = \cancel{-x^5} \cancel{+ 10x^4} \cancel{- 80x} \cancel{+ 32}$$

$$80x^2 - 40x^3 = 0$$

[1]

Factorising gives

$$40x^2(2 - x) = 0$$

[1]

Finding the solutions to the equation

$x = 0, x = 2$ [1]

(4 marks)

2 In the expansion of $\left(2k - \frac{x}{k}\right)^5$, where k is a constant, the coefficient of x^2 is 160.

Find the value of k.

Answer

We can deduce that the term containing x^2 is

$$^5C_2(2k)^3\left(-\frac{x}{k}\right)^2$$

[1]

Simplify this

$$= 10 \times 8k^3 \times \frac{x^2}{k^2}$$

$$= 80kx^2$$

The coefficient of x^2 is equal to 160 therefore

$$80k = 160$$

[1]

Divide both sides by 80 to get the final answer

$k = 2$ [1]

(3 marks)

3 (i) Find the first 3 terms in the expansion of $(1 + 3x)^6$, in ascending powers of x. Simplify the coefficient of each term.

(ii) When the expansion of $(1 + 3x)^6(a + x)^2$ is written in ascending powers of x, the first three terms are $4 + 68x + bx^2$, where a and b are constants. Find the value of a and the value of b.

Answer

i) Use binomial expansion to obtain the first three terms

$$(1+3x)^6 = 1^6 + {}^6C_1 \times 1^5 \times (3x)^1 + {}^6C_2 \times 1^4 \times (3x)^2 + \ldots$$

an unsimplified correct substitution or any two correct final terms from the answer below [1]

Simplify

$$= 1 + 6 \times 3x + 15 \times 9x^2$$

$$\mathbf{1 + 18x + 135x^2}$$ [1]

ii) The expansion of $(a+x)^2$ is $a^2 + 2ax + x^2$. Multiply the first three terms of $(1+3x)^6$ found in (i) by $a^2 + 2ax + x^2$, up to terms in x^2 (you don't need to multiply terms that will result with x^3 or higher)

$$(1 + 18x + 135x^2 + \ldots)(a^2 + 2ax + x^2)$$
$$= (a^2)(1) + (a^2)(18x) + (a^2)(135x^2) + \ldots + (2ax)(1) + (2ax)(18x) + \ldots + (x^2)(1) + \ldots$$

(The line above is really just shown for demonstration purposes. In reality we'd probably go straight to the line below)

$$(1 + 18x + 135x^2 + \ldots)(a^2 + 2ax + x^2) = a^2 + 18a^2x + 135a^2x^2 + \ldots + 2ax + 36ax^2 + \ldots + x^2 + \ldots$$

We can equate this to $4 + 68x + bx^2$

$$a^2 + 18a^2x + 135a^2x^2 + \ldots + 2ax + 36ax^2 + \ldots + x^2 + \ldots = 4 + 68x + bx^2$$

The only constant term on the left hand side is "a^2" therefore equating the constants gives

$$a^2 = 4$$
$$a = \pm 2$$

[1]

Only one of these can be correct. Examining the terms in x:

$$a = -2, \quad 18a^2x + 2ax = 18(-2)^2x + 2(-2)x = 68x$$
$$a = 2, \quad 18a^2x + 2ax = 18(2)^2x + 2(2)x = 76x$$

But we know that the term in x is $68x$ therefore $a \neq 2$ and

$$\mathbf{a = -2}$$ [1]

Now substitute $a = -2$ into the terms in x^2

$$135(-2)^2x^2 + 36(-2)x^2 + x^2 = bx^2$$
$$540x^2 - 72x^2 + x = bx^2$$
$$469x^2 = bx^2$$

Therefore

$b = 469$ [1]

(5 marks)

4 The first 3 terms in the expansion of $(3 - ax)^5$, in ascending powers of x, can be written in the form $b - 81x + cx^2$. Find the value of each of a, b and c.

Answer

Use the binomial theorem $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n$

$$(3 - ax)^5 = 3^5 + \binom{5}{1}3^4(-ax) + \binom{5}{2}3^3(-ax)^2 + \ldots$$

$$(3 - ax)^5 = 243 - 405ax + 270a^2x^2 + \ldots$$

1 mark for second and third term [2]

Set the first 3 terms equal to the expression given in the question:

$$b - 81x + cx^2 = 243 - 405ax + 270a^2x^2$$

Equate coefficients:

$$b = 243$$

[1]

$$-81 = -405a \text{ so } a = \frac{1}{5}$$

[1]

$$c = 270a^2 = 270 \times \left(\frac{1}{5}\right)^2 = \frac{54}{5}$$

$a = \frac{1}{5}$, $b = 243$, $c = \frac{54}{5}$ [1]

(5 marks)

5 (a) Find the first 3 terms in the expansion of $\left(2+\frac{k}{2}\right)^8$ in ascending powers of k.

Simplify the coefficient of each term.

Answer

Substitute $a=2,\ b=\frac{k}{2}$ and $n=8$ into the first three terms of the binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

$$\left(2+\frac{k}{2}\right)^8 = 2^8 + \binom{8}{1}2^7\left(\frac{k}{2}\right) + \binom{8}{2}2^6\left(\frac{k}{2}\right)^2 + \ldots$$

Simplify each term

$$\left(2+\frac{k}{2}\right)^8 = 256 + 8\times128\left(\frac{k}{2}\right) + 28\times64\left(\frac{k^2}{4}\right) + \ldots$$

$256 + 512k + 448k^2$

[B1 B1]

Mark Scheme and Guidance

B1: For at least two terms correct.

B1: For the fully correct final answer.

(2 marks)

(b) Hence find the first three terms in ascending powers of p in the expansion of $\left(2+\frac{p-p^2}{2}\right)^8$.

Simplify the coefficient of each term.

Answer

Substitute $k=p-p^2$ into the answer in part (a)

$$256 + 512(p-p^2) + 448(p-p^2)^2$$

Expand

$$256 + 512p - 512p^2 + 448(p^2 - 2p^3 + p^4)$$

Ignore any terms in p^3 or higher

$$256 + 512p - 512p^2 + 448p^2$$

Collect like terms

$256 + 512p - 64p^2$

[B1 B1]

Mark Scheme and Guidance

B1: For at least two terms correct.

B1: For the fully correct final answer.

(2 marks)

(c) Find the term independent of x in the expansion of $\left(ax^2 - \frac{1}{x^2}\right)^6$ where $a \neq 0$.

Give your answer in terms of a.

Answer

Substitute $a = ax^2$, $b = \left(-\frac{1}{x^2}\right)$ and $n = 6$ into the formula for a general term $\binom{n}{r}a^{n-r}b^r$

$$\binom{6}{r}(ax^2)^{6-r}\left(-\frac{1}{x^2}\right)^r$$

[M1]

Rewrite this expression using index laws, noting that $\left(-\frac{1}{x^2}\right) = (-1)x^{-2}$

$$\binom{6}{r}a^{6-r}x^{2(6-r)}(-1)^r x^{-2r}$$

Collect the terms in x using index laws

$$\binom{6}{r}a^{6-r}(-1)^r x^{2(6-r)-2r}$$

The term 'independent of x' is the term without an x, i.e. $x^0 = 1$

Set the power of x equal to zero and solve

$$\begin{aligned} 2(6-r)-2r &= 0 \\ 12-2r-2r &= 0 \\ 12 &= 4r \\ r &= 3 \end{aligned}$$

[A1]

Substitute $r = 3$ back into the general term

$$= \binom{6}{3}a^{6-3}(-1)^3 x^{2(6-3)-2\times 3}$$

$$= \binom{6}{3}a^3(-1)^3 x^0$$

$$= -\binom{6}{3}a^3$$

Work out $\binom{6}{3}$, e.g. on your calculator

$$\binom{6}{3} = 20$$

Substitute this back into the term to get the final answer

$-20a^3$

[A1]

(3 marks)

Very Hard Questions

1 The first 3 terms in the expansion of $(a+x)^3\left(1-\frac{x}{3}\right)^5$, in ascending powers of x, can be written in the form $27+bx+cx^2$, where a, b and c are integers. Find the values of a, b and c.

Answer

Expand both the brackets up to their x^2 terms using the Binomial Expansion. This will lead us to form $27+bx+cx^2$.

$$(a+x)^3=(1)(a)^3+\left({}^3C_1\right)(a)^2(x)+\left({}^3C_2\right)(a)(x)^2+\ \dots$$

[1]

$$\left(1-\frac{x}{3}\right)^5=(1)(1)^5+\left({}^5C_1\right)(1)^4\left(-\frac{x}{3}\right)+\left({}^5C_2\right)(1)^3\left(-\frac{x}{3}\right)^2+\ \dots$$

$$=1+(5)\left(-\frac{x}{3}\right)+(10)\left(\frac{x^2}{9}\right)+\ \dots$$

$$=1-\frac{5}{3}x+\frac{10}{9}x^2+\ \dots$$

[2]

Multiply together.

$$(a^3+3a^2x+3ax^2+\dots)\left(1-\frac{5}{3}x+\frac{10}{9}x^2+\dots\right)$$

Find the constant term and equate to 27.

$$a^3=27$$

$\boldsymbol{a=3}$ [1]

Substitute $a=3$ into the expression.

$$=((3)^3+(3)(3)^2x+(3)(3)x^2+\dots)\left(1-\frac{5}{3}x+\frac{10}{9}x^2+\dots\right)$$

$$=(27+27x+9x^2+\dots)\left(1-\frac{5}{3}x+\frac{10}{9}x^2+\dots\right)$$

Find the coefficient of x and equate to b.

$$b = 27 - \frac{5}{3}(27)$$

[1]

$b = -18$ [1]

Find the coefficient of x^2 and equate to c.

$$c = 9 + (27)\left(\frac{10}{9}\right) + (27)\left(-\frac{5}{3}\right)$$

[1]

$c = -6$ [1]

(8 marks)

2 The first three terms in the expansion of $(a + bx)^5 (1 + x)$ are $32 - 208x + cx^2$. Find the value of each of the integers a, b and c.

Answer

Using the formula for the binomial expansion, finding the first three terms gives

$$\left(a^5 + \binom{5}{1} a^4 bx + \binom{5}{2} a^3 (bx)^2\right)(1 + x)$$

[1]

Simplifying gives

$$(a^5 + 5a^4 bx + 10a^3 b^2 x^2)(1 + x)$$

[1]

Expanding the brackets

$$a^5 + 5a^4 bx + 10a^3 b^2 x^2 + a^5 x + 5a^4 bx^2 + 10a^3 b^2 x^3$$

multiplying out brackets [1]

all terms correct [1]

Comparing to $32 - 208x + cx^2$ by comparing the constants we have

$$a^5 = 32$$

$a = 2$ [1]

Equating the coefficients of x we have

$$5a^4b + a^5 = -208$$

Substituting in $a = 2$

$$80b + 32 = -208$$

$$80b = -240$$

$b = -3$ [1]

Equating the coefficients of x^2 we have

$$10a^3b^2 + 5a^4b = c$$

Substituting in $a = 2$ and $b = -3$ we have

$$720 - 240 = c$$

$c = 480$ [1]

(7 marks)

3 Given that the coefficient of x^2 in the expansion of $(1+x)\left(1-\frac{x}{2}\right)^n$ is $\frac{25}{4}$, find the value of the positive integer n.

Answer

Expand $\left(1-\frac{x}{2}\right)^n$ using the binomial expansion up to the x^2 term.

$$(1)^n + \binom{n}{1}(1)^{n-1}\left(-\frac{x}{2}\right) + \binom{n}{2}(1)^{n-2}\left(-\frac{x}{2}\right)^2 + \ldots$$

$$1 + \left(\frac{n!}{(n-1)!1!}\right)\left(-\frac{x}{2}\right) + \left(\frac{n!}{(n-2)!\,2!}\right)\left(-\frac{x}{2}\right)^2 + \ldots$$

$$1 + (n)\left(-\frac{x}{2}\right) + \left(\frac{n(n-1)}{2}\right)\left(\frac{x^2}{4}\right) + \ldots$$

Multiplied by $(1+x)$, this is:

$$(1+x)\left(1 + (n)\left(-\frac{x}{2}\right) + \left(\frac{n(n-1)}{2}\right)\left(\frac{x^2}{4}\right) + \ldots\right)$$

for the correct x term [1]

for the correct x^2 term [1]

When expanded, the x^2 terms will be

$$1 \times \left(\frac{n(n-1)}{2}\right)\left(\frac{x^2}{4}\right) \text{ and } x \times \left(n\left(-\frac{x}{2}\right)\right)$$

Simplify.

$$\left(\frac{n(n-1)}{8}\right)x^2 \text{ and } \left(-\frac{n}{2}\right)x^2$$

Sum the coefficients of x^2 and equate to $\frac{25}{4}$

$$\left(\frac{n(n-1)}{8}\right) + \left(-\frac{n}{2}\right) = \frac{25}{4}$$

[1]

Expand the brackets.

$$\frac{n^2-n}{8} - \frac{n}{2} = \frac{25}{4}$$

Multiply through by 8.

$$n^2 - n - 4n = 50$$

Rearrange and solve.

$$n^2 - 5n - 50 = 0$$
$$(n-10)(n+5) = 0$$

$$n = 10 \text{ and } n = -5$$

We are told in the question that the value of n is positive. Therefore:

$n = 10$ [1]

(5 marks)

4 In the expansion of $\left(1+\frac{x}{2}\right)^n$ the coefficient of x^4 is half the coefficient of x^6.

Find the value of the positive constant n.

Answer

Find an expression for the coefficient of the x^4 term of the expansion.

$$\frac{n!}{4!(n-4)!}\times(1)^{n-4}\times\left(\frac{1}{2}\right)^4$$

This can be written as

$$\frac{n(n-1)(n-2)(n-3)\times(n-4)!}{4!\times(n-4)!}\times\left(\frac{1}{2}\right)^4$$

Simplify.

$$\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}\times\left(\frac{1}{2}\right)^4$$

[1]

Similarly, find an expression for the x^6 coefficient of the term of the expansion.

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\times2\times3\times4\times5\times6}\times\left(\frac{1}{2}\right)^6$$

[1]

We are told that the coefficient of x^4 is half the coefficient of x^6, therefore:

$$\frac{n(n-1)(n-2)(n-3)}{24\times16}=\frac{1}{2}\times\left(\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720\times64}\right)$$

[1]

Simplify.

$$\frac{2 \times 720 \times 64}{24 \times 16} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{n(n-1)(n-2)(n-3)}$$

$$\frac{720}{3} = (n-4)(n-5)$$

$$240 = (n-4)(n-5)$$

[1]

Expand the brackets, rearrange and solve.

$$n^2 - 9n + 20 = 240$$

$$n^2 - 9n - 220 = 0$$

$$(n-20)(n+11) = 0$$

[1]

$$n = 20 \text{ or } n = -11$$

The question asks for the positive value.

$n = 20$ [1]

(6 marks)