IGCSE · Cambridge (CIE) · Further Maths   1 hour   11 questions

Exam Questions

Circular Measure

Radian Measure / Arcs & Sectors

Medium (3 questions)	/15
Hard (4 questions)	/30
Very Hard (4 questions)	/34
Total Marks	**/79**

Medium Questions

1 A circle has a radius of 6 cm. A sector of this circle has a perimeter of $2(6+5\pi)$ cm. Find the area of this sector.

Answer

The circle has radius = 6cm

Calculate the length of the arc

$$\text{arc length} = 2(6+5\pi) - (2\times 6)$$

$$\text{arc length} = 12 + 10\pi - 12$$

$$\text{arc length} = 10\pi$$

[1]

Calculate the circumference of the circle

$$C = 2\times\pi\times 6$$

$$C = 12\pi$$

Work out the fraction of the area that we want

$$\frac{10\pi}{12\pi} = \frac{5}{6}$$

[1]

Calculate the area of the sector

$$\text{area of sector} = \frac{5}{6}\times\pi\times 6^2$$

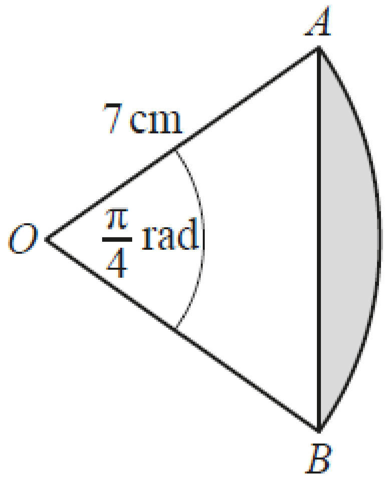
[1]

$$\text{area of sector} = 30\pi$$

$30\pi\ \text{cm}^2$ [1]

(4 marks)

2



The diagram shows the sector AOB of a circle with centre O and radius 7 cm.

Angle $AOB = \dfrac{\pi}{4}$ radians. Find the perimeter of the shaded region.

Answer

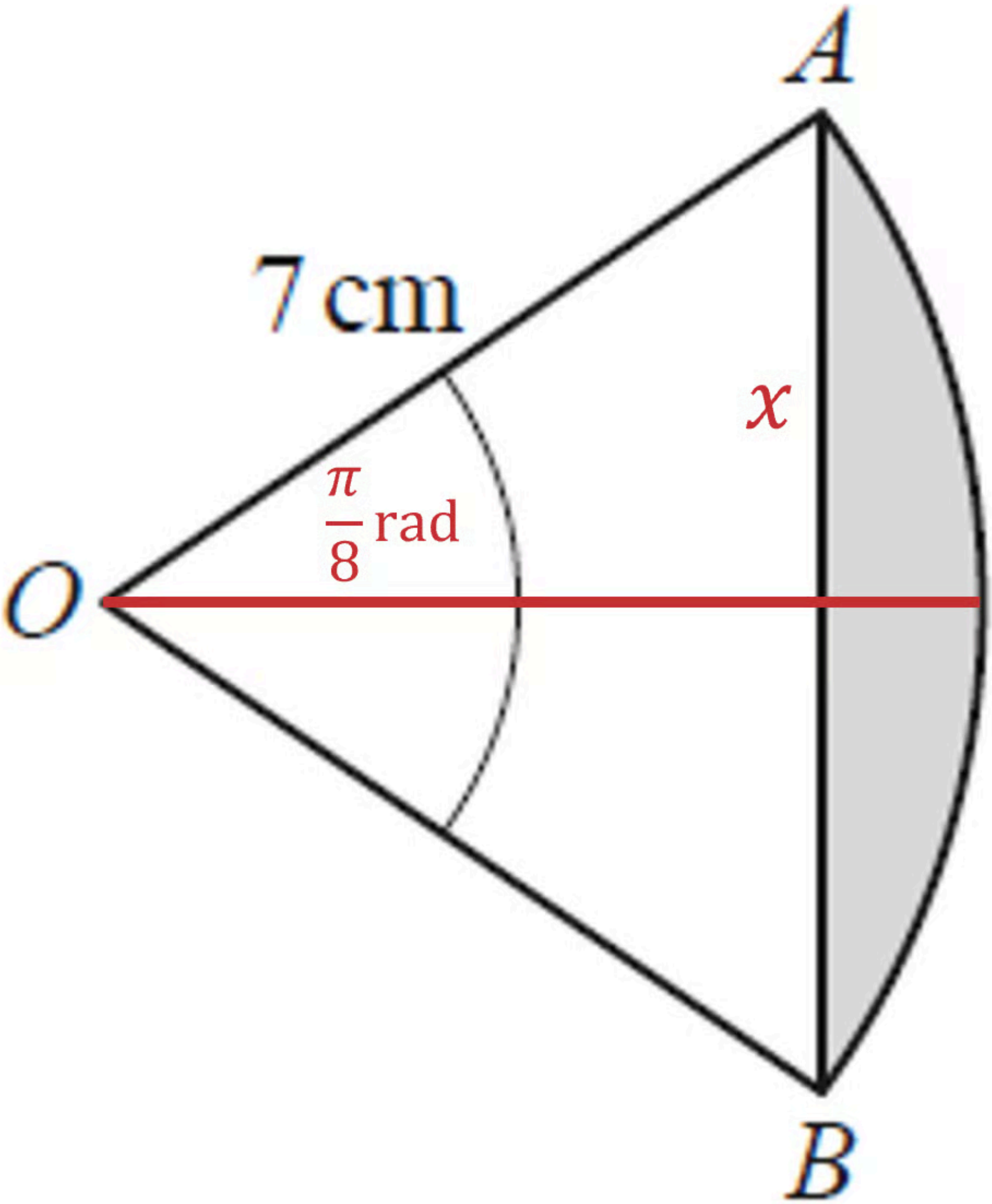
Use the formula arc length $= r\theta$ to find the length of the arc

$$\text{arc length} = 7 \times \frac{\pi}{4}$$

$$\text{arc length} = \frac{7\pi}{4}$$

Use trigonometry to find the length of the chord

Bisect *AB* and draw a right-angled triangle



Use the sin ratio to find length x

$$\sin\left(\frac{\pi}{8}\right) = \frac{x}{7}$$

$$x = 7\sin\left(\frac{\pi}{8}\right)$$

The length of the chord is double this so multiply by 2

$$\text{chord length} = 14\sin\left(\frac{\pi}{8}\right)$$

[1]

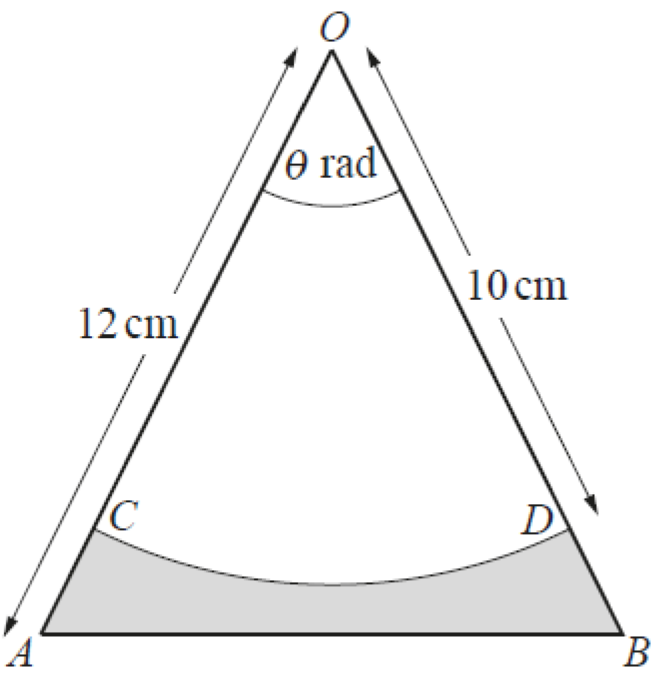
The perimeter of the shaded region is the arc length added to the chord length

$$\text{perimeter of shaded region} = 14\sin\left(\frac{\pi}{8}\right) + \frac{7\pi}{4}$$

[1]

10.9 cm [1]
(3 marks)

3 (a)



The diagram shows an isosceles triangle OAB such that $OA = OB = 12\text{cm}$ and angle $AOB = \theta$ radians.
Points C and D lie on OA and OB respectively such that CD is an arc of the circle, centre

O, radius 10 cm.
The area of the sector $OCD = 35\text{cm}^2$.

Show that $\theta = 0.7$.

Answer

In radians, Area of sector $= \dfrac{1}{2} r^2 \theta$

Substitute known values into the equation.

$$35 = \frac{1}{2}(10)^2 \times \theta$$

Rearrange and solve for θ.

$$35 = 50 \times \theta$$
$$\theta = \frac{35}{50}$$
$$\theta = 0.7$$

0.7 [1]
(1 mark)

(b) Find the perimeter of the shaded region.

Answer

Find the arc length CD.

$$CD = r\theta$$
$$CD = 10 \times 0.7$$
$$CD = 7\,cm$$

[1]

Work out length $\dfrac{AB}{2}$ by using trigonometry.

$$\sin(0.35) = \frac{AB/2}{12}$$

$$\frac{AB}{2} = 4.11477$$

[1]

Double to find AB.

$$AB = 8.229547cm$$

[1]

Add the sides to find the perimeter.

$$\text{Perimeter} = 7 + 8.23 + 2 + 2$$

19.2 cm [1]

(4 marks)

(c) Find the area of the shaded region.

Answer

Area of shaded region = Area of triangle – Area of sector

We are told the area of the sector is $35cm^2$.

Find the area of the triangle *AOB.*

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(12)^2 \times \sin(0.7)$$

[1]

$$\text{Area} = 46.38$$

[1]

Subtract the area of the sector from the area of the triangle.

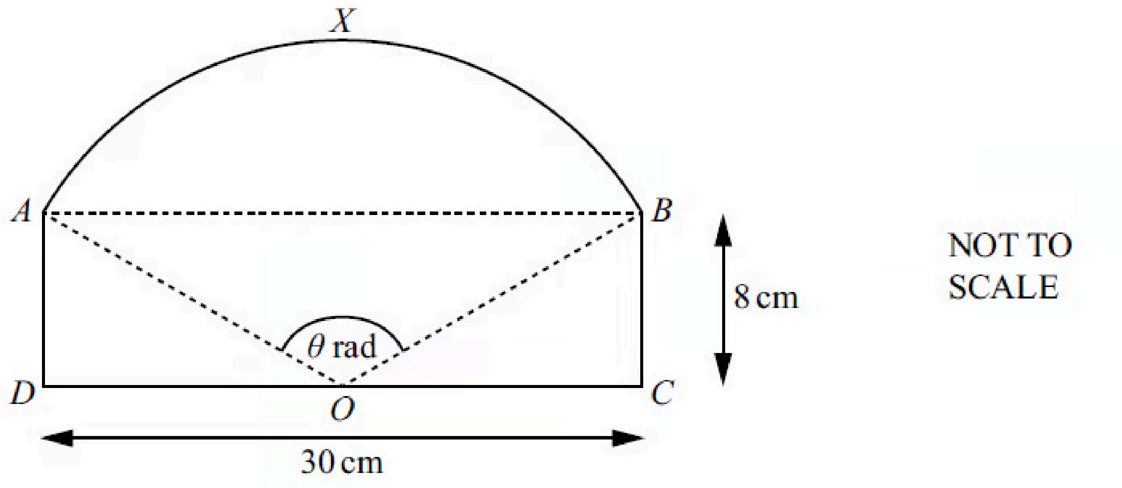
$$46.38 - 35 = 11.38$$

$11.4\ \text{cm}^2$ [1]

(3 marks)

Hard Questions

1 (a)



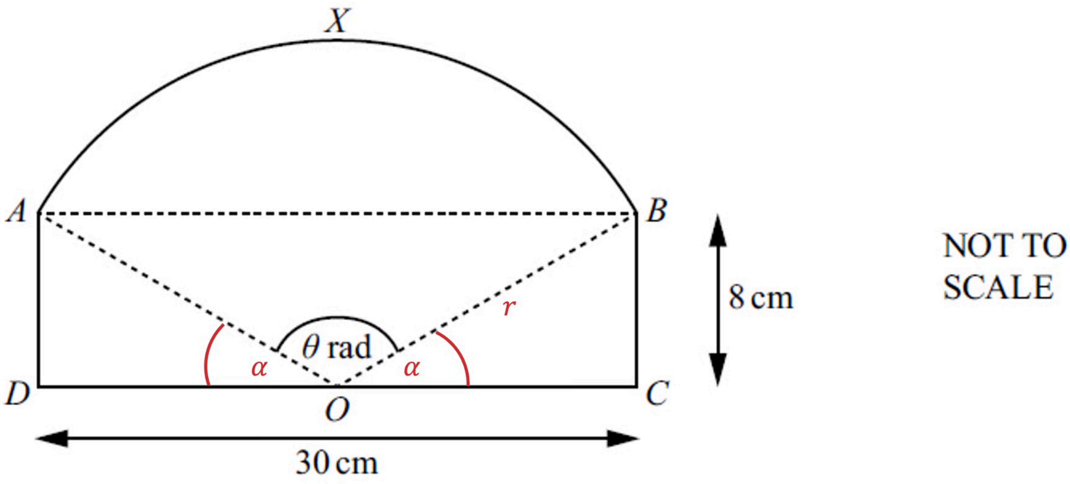
The diagram shows a rectangle $ABCD$ and an arc AXB of a circle with centre at O, the midpoint of DC.
The length of BC is 8 cm and the length of DC is 30 cm. Angle AOB is θ radians.

Find the perimeter of the shape $ADOCBX$.

Answer

Let's label the radius of the sector as r and angles BOC and DOC as α.

$DO = OC = \frac{30}{2} = 15$ cm



We can find r using Pythagoras theorem

$$r = \sqrt{8^2 + 15^2}$$

[1]

$$= 17$$

We can find α using right-angle trigonometry. θ is in radians so remember to set your calculator to radians

$$\alpha = \tan^{-1}\left(\frac{8}{15}\right) = 0.489957326...$$

Now find θ by subtracting 2α from π

$$\theta = \pi - 2(0.489957326...)$$

[1]

$$= 2.16167800...$$

[1]

(It's worth saving this result in your calculator as it will be used more than once in this question)

Now find the length of the arc using arc length = θr (or $\frac{\theta}{2\pi}(2\pi r)$)

arc length = $2.16167800... \times 17$

And finally find the perimeter of $ADOCBX$ by adding the 3 straight lengths to the arc length

$$\text{perimeter} = 8 + 8 + 30 + 2.16167800... \times 17$$

[1]

$$= 82.748560...$$

82.7 cm (3 s.f.) [1]
(5 marks)

(b) Find the area of the shape $ADOCBX$.

Answer

We know from part (a) that $\theta = 2.16167800...$

Find the area of the sector *OAXB*

$$\frac{1}{2} \times 17^2 \times 2.16167800... = 312.362...$$

Find the area of the triangle *OBC* (and *OAD*)

$$\frac{1}{2} \times 15 \times 8 = 60$$

Add together the area of the sector and the area of the two triangles

$$\text{Total area} = \frac{1}{2} \times 17^2 \times 2.16167800... + 60 + 60$$
$$= 432.362471...$$

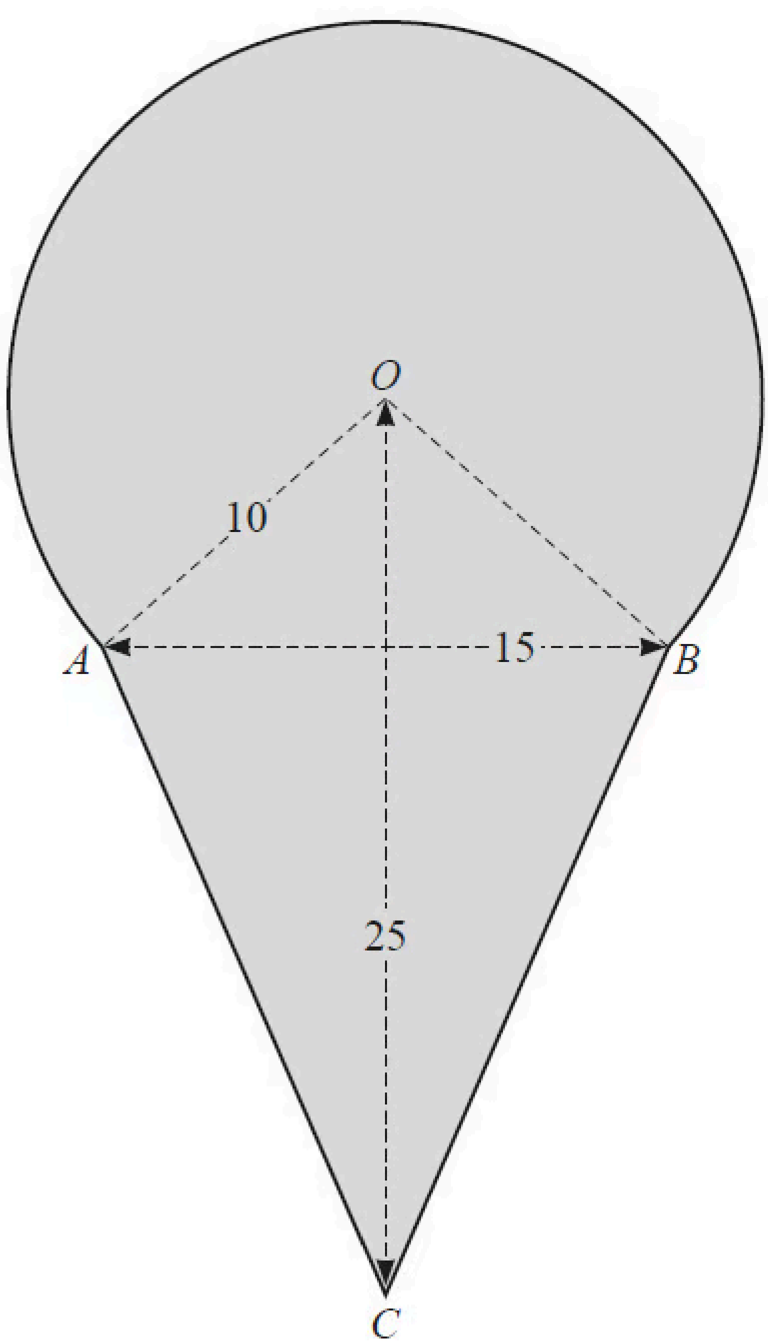
[1]

Round the final answer to at least 3 significant figures

432 cm^2 (3 s.f.) [1]

(2 marks)

2 (a) In this question all lengths are in centimetres.

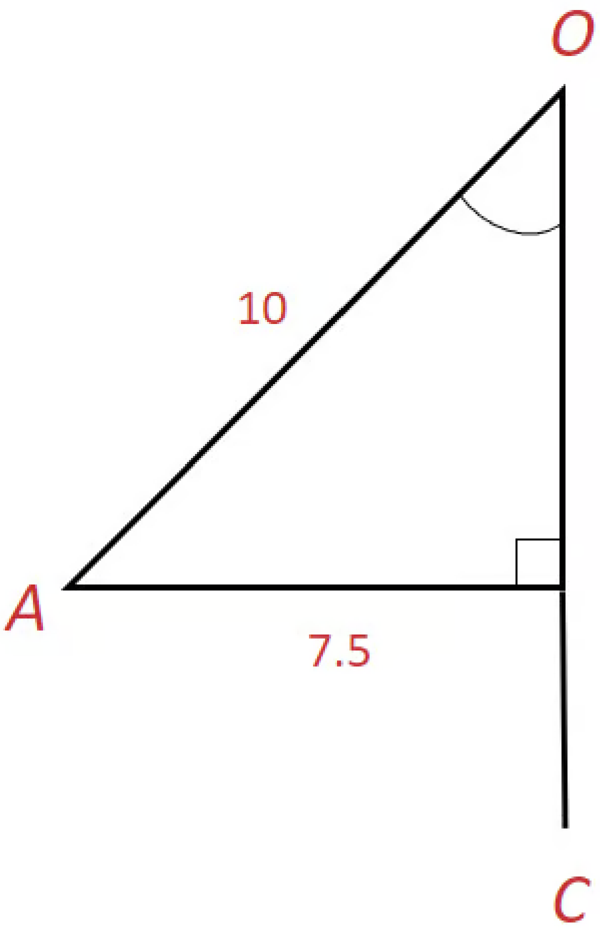


The diagram shows a shaded shape. The arc AB is the major arc of a circle, centre O, radius 10. The line AB is of length 15, the line OC is of length 25 and the lengths of AC and BC are equal.

Show that the angle AOB is 1.70 radians correct to 2 decimal places.

Answer

Find angle *AOC* using trigonometry.



$$\text{Angle } AOC = \sin^{-1}\left(\frac{7.5}{10}\right)$$

[1]

$$\text{Angle } AOC = 0.84806$$

Angle *AOB* is double angle *AOC*.

$$\text{Angle } AOB = 2 \times 0.848$$

$$\text{Angle } AOB = 1.696$$

$AOB = 1.70 \text{ to 2dp}$ [1]

(2 marks)

(b) Find the perimeter of the shaded shape.

Answer

Use the Cosine rule to work out the length *AC*.

$$AC^2 = 10^2 + 25^2 - (2 \times 10 \times 25 \times \cos(AOC))$$
$$= 725 - (500 \times \cos(0.848))$$

[1]

$$AC^2 = 394.281$$

$$AC = 19.857$$

[1]

Work out the length of major arc *AB*.

$$\pi d \times \frac{\text{angle of sector}}{2\pi}$$

$$20\pi \times \frac{(2\pi - 1.7)}{2\pi} = 45.832$$

[1]

Add together the length of major arc *AB*, and $2 \times$ length *AC* (since *AC* = *BC*).

$$45.832 + (2 \times 19.857) = 85.546$$

$= 85.55$ to 2dp [1]

answers which round to 85.5 or 85.6 are accepted

(4 marks)

(c) Find the area of the shaded shape.

Answer

Find the area of major sector AOB.

$$\pi r^2 \times \frac{\text{angle of sector}}{2\pi}$$

$$\pi \times (10)^2 \times \frac{(2\pi - 1.7)}{2\pi}$$

[1]

$$= 229.159$$

[1]

Find the area of kite AOBC.

$$\frac{15 \times 25}{2} = 187.5$$

[1]

Add together the area of the major sector and the area of the kite.

$$229.159 + 187.5$$

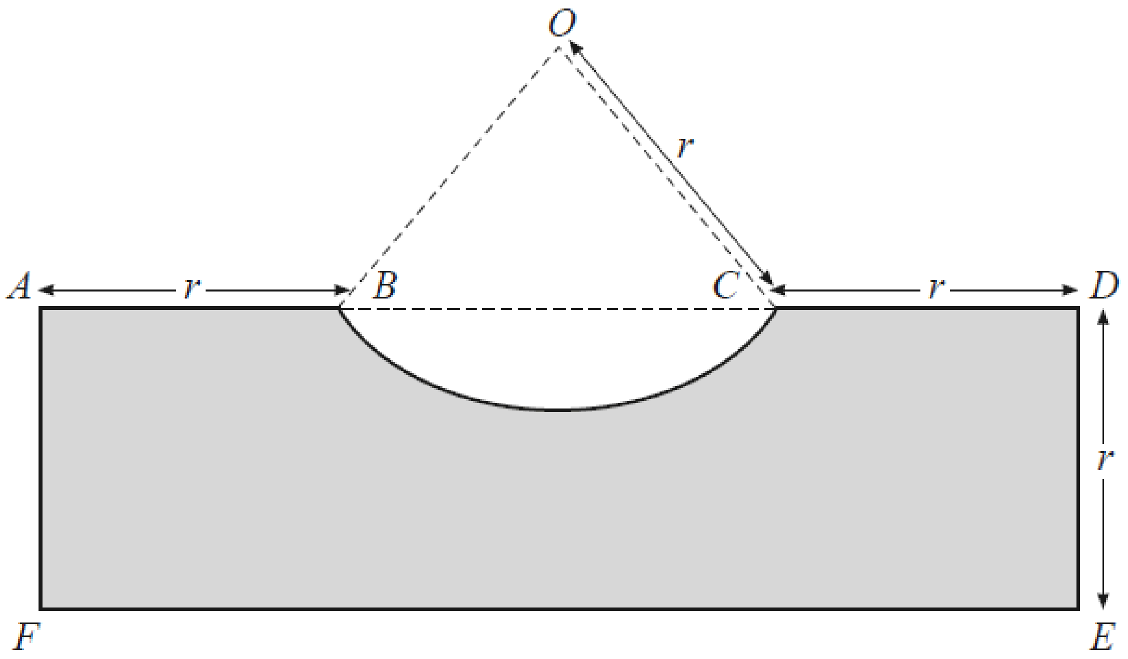
[1]

416.659 [1]

answers which round to 417 are accepted

(5 marks)

3 (a) In this question all lengths are in centimetres and all angles are in radians.



The diagram shows the rectangle $ADEF$, where $AF = DE = r$. The points B and C lie on AD such that $AB = CD = r$. The curve BC is an arc of the circle, centre O, radius r and has a length of $1.5r$.

Show that the perimeter of the shaded region is $(7.5 + 2\sin 0.75)r$.

Answer

Since arc length, in radians, is $r\theta$ and we are told that the arc length BC is $1.5r$

$$\text{Angle } BOC = 1.5 \text{ radians}$$

[1]

Use trigonometry to find an expression for the length BC in terms of r.

$$\sin(0.75) = \frac{BC/2}{r}$$

[1]

Rearrange to make BC the subject.

$$BC = 2r \times \sin(0.75)$$

[1]

Add together the lengths around the perimeter of the shaded area.

$$2r + 4r + 1.5r + (2r\sin(0.75))$$

[1]

Simplify.

$$7.5r + 2r\sin(0.75)$$

Factorise by pulling out a factor of r.

$(7.5 + 2\sin 0.75)r$ [1]

(5 marks)

(b) Find the area of the shaded region, giving your answer in the form kr^2, where k is a constant correct to 2 decimal places.

Answer

Find the area of the rectangle by adding together the lengths *AB*, *BC* and *CD*, and multiplying it by *DE*.

$$\text{Area (rectangle)} = (r + 2r\sin 0.75 + r) \times r$$

Simplify.

$$= (2r + 2r\sin 0.75)r$$

[1]

Write in terms of r^2

$$= 2r^2 + 2r^2(\sin 0.75)$$
$$= r^2(2 + 2\sin 0.75)$$
$$= 3.36r^2$$

Find the area of the segment by subtracting the area of triangle *BOC* from the area of the sector.

$$\text{Area (sector)} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}r^2(1.5)$$

$$\text{Area (triangle)} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}r^2(\sin 1.5)$$

$$\text{Area (segment)} = \frac{1}{2}r^2(1.5) - \frac{1}{2}r^2(\sin 1.5)$$

Factorise by pulling out a factor of $\frac{1}{2}r^2$ and simplify to write in terms of r^2

$$\text{Area (segment)} = \frac{1}{2}r^2(1.5 - \sin 1.5)$$
$$= 0.251r^2$$

[1]

Subtract the area of the segment from the area of the rectangle.

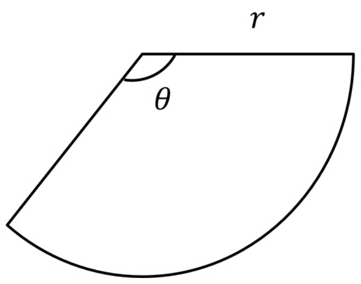
$$\text{Area (shaded)} = 3.36r^2 - 0.251r^2$$

[1]

$3.11r^2$ [1]

(4 marks)

4 A sector from a circle of radius r has an internal angle of θ radians, as shown below.



The perimeter of the sector is 4 units and the area of the sector is A square units.

Show that $A = \dfrac{8\theta}{(2+\theta)^2}$.

Answer

The formula for the length of the arc in radians is $r\theta$

Find an expression for the total perimeter and set it equal to 4

$$r + r\theta + r = 4$$

$$2r + r\theta = 4 \quad (1)$$

[B1]

Mark Scheme and Guidance

This mark is for any correct perimeter equation that uses 4 and $r\theta$.

The formula for the area of the arc in radians is $\frac{1}{2}r^2\theta$

Find an expression for the area A

$$A = \frac{1}{2}r^2\theta \qquad (2)$$

Eliminate r, e.g. make r the subject of equation 1 by first factorising it out

$$r(2+\theta) = 4$$

$$r = \frac{4}{2+\theta}$$

[M1]

Then substitute this into equation 2

$$A = \frac{1}{2}\left(\frac{4}{2+\theta}\right)^2 \times \theta$$

Simplify

$$A = \frac{1}{2} \times \frac{16}{(2+\theta)^2} \times \theta$$

$$A = \frac{8\theta}{(2+\theta)^2}$$

[A1]

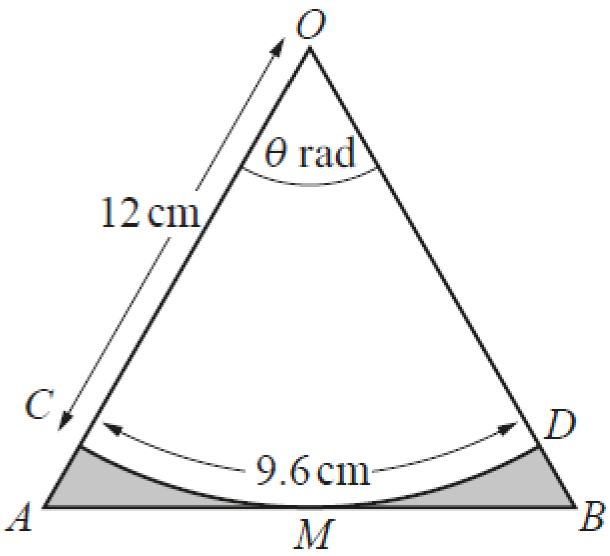
Mark Scheme and Guidance

The last mark is for showing the steps that get you from the substitution of r into A to the answer given (not for writing out the "show that" answer).

(3 marks)

Very Hard Questions

1 (a)



The diagram shows an isosceles triangle OAB such that $OA = OB$ and angle $AOB = \theta$ radians. The points C and D lie on OA and OB respectively. CD is an arc of length 9.6 cm of the circle, centre O, radius 12 cm. The arc CD touches the line AB at the point M.

Find the value of θ.

Answer

Using the formula $l = r\theta$

$$9.6 = 12\theta$$

Solve to find θ

$$\frac{9.6}{12} = \theta$$

0.8 [1]

(1 mark)

(b) Find the total area of the shaded regions.

Answer

Using the formula for the area of a sector $A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2} \times 12^2 \times 0.8 = 57.6 \text{ cm}^2$$

[1]

Calculating the length of the base of the right-angled triangle OAM, using the fact that the length of the radius would also be the length of the perpendicular height, and the angle is half of 0.8,

$$\tan(0.4) = \frac{AM}{12}$$

$$AM = 5.074$$

[1]

Therefore, $AO = 2 \times 5.074 = 10.148$

Calculating the area of triangle OAB

$$\frac{1}{2}(10.148) \times 12 = 60.88$$

[1]

Subtracting the area of the sector from the area of the triangle

$$60.88 - 57.6$$

3.28 cm^2 [1]

(4 marks)

(c) Find the total perimeter of the shaded regions.

Answer

Calculating the length of OA

$$\cos(0.4) = \frac{12}{OA}$$

[1]

$$OA = 13.028$$

Subtracting OC from OA gives

$$AC = 1.028$$

Finding the perimeter of the shaded region

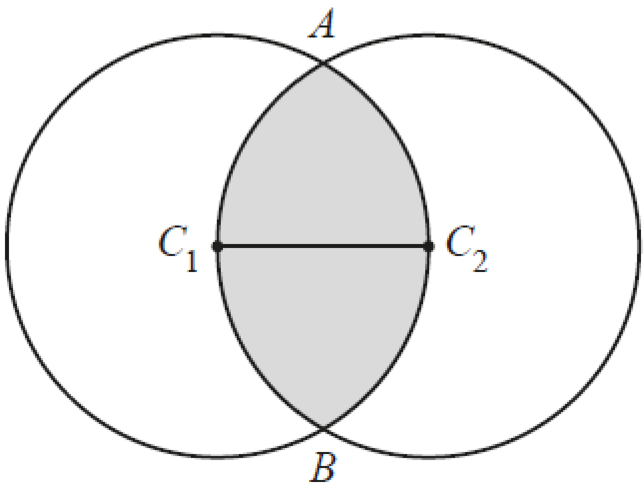
$$\text{Perimeter} = 1.028 + 1.028 + 9.6 + 5.07 + 5.07$$

[1]

$21.8\ cm$ [1]

(3 marks)

2 (a)



The circles with centres C_1 and C_2 have equal radii of length r cm. The line C_1C_2 is a radius of both circles. The two circles intersect at A and B.

Given that the perimeter of the shaded region is 4π cm, find the value of r. [4]

Answer

The total perimeter is 4π so the perimeter of one arc AB is 2π

Find the angle AC_1C_2

AC_1 is a radius and AC_2 is also a radius, and C_1C_2 is a radius too meaning that triangle AC_1C_2 is an equilateral triangle so

$$\text{angle } AC_1C_2 = 60° = \frac{\pi}{3}$$

Use this to find angle ACB

$$\text{angle } ACB = 2 \times \frac{\pi}{3}$$

$$\text{angle } ACB = \frac{2\pi}{3}$$

[1]

Apply the formula for arc length using half of the perimeter

$$\text{arc length} = r\theta$$

$$2\pi = r \times \frac{2\pi}{3}$$

correct arc length 2π [1]

correct equation [1]

Multiply by 3

$$2\pi r = 6\pi$$

Divide by 2π

$$r = 3$$

$r = 3$ [1]

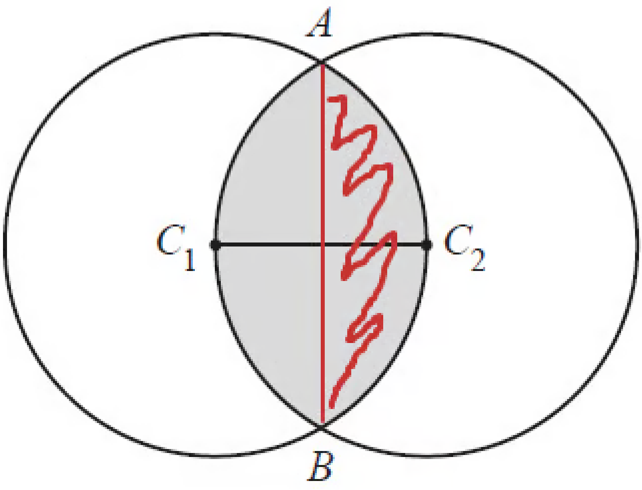
(4 marks)

(b) Find the exact area of the shaded region.

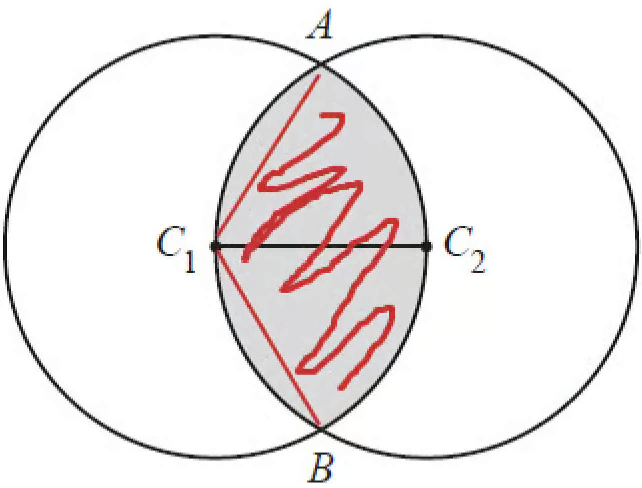
Answer

From part (a) $r = 3$ and angle $ACB = \frac{2\pi}{3}$

Split the area in half by drawing a line connecting A and B



Find the red shaded area by first working out the area of the sector below



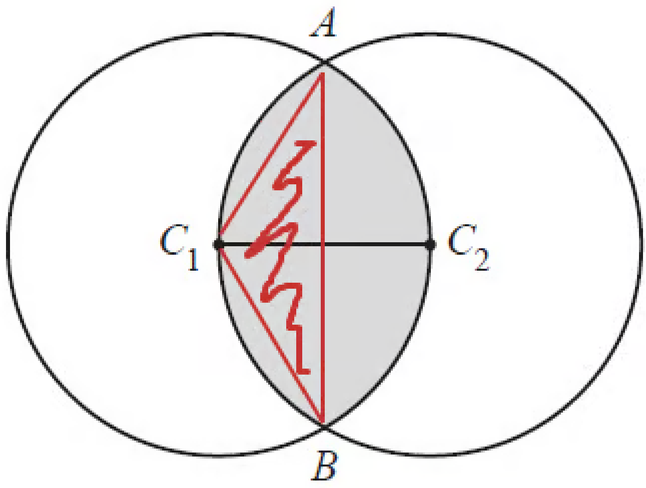
Use the area of a sector formula, area of sector $= \frac{1}{2} r^2 \theta$

$$\text{area of sector } AC_1B = \frac{1}{2} \times 3^2 \times \frac{2\pi}{3}$$

[1]

$$\text{area of sector } AC_1B = 3\pi$$

Now find the area of the triangle using area $= \frac{1}{2}ab\sin C$



$$\text{area of triangle} = \frac{1}{2} \times 3 \times 3 \times \sin\left(\frac{2\pi}{3}\right)$$

[1]

$$\text{area of triangle} = \frac{9\sqrt{3}}{4}$$

Subtract the triangle from the sector to get the shaded area in the first diagram

$$3\pi - \frac{9\sqrt{3}}{4}$$

[1]

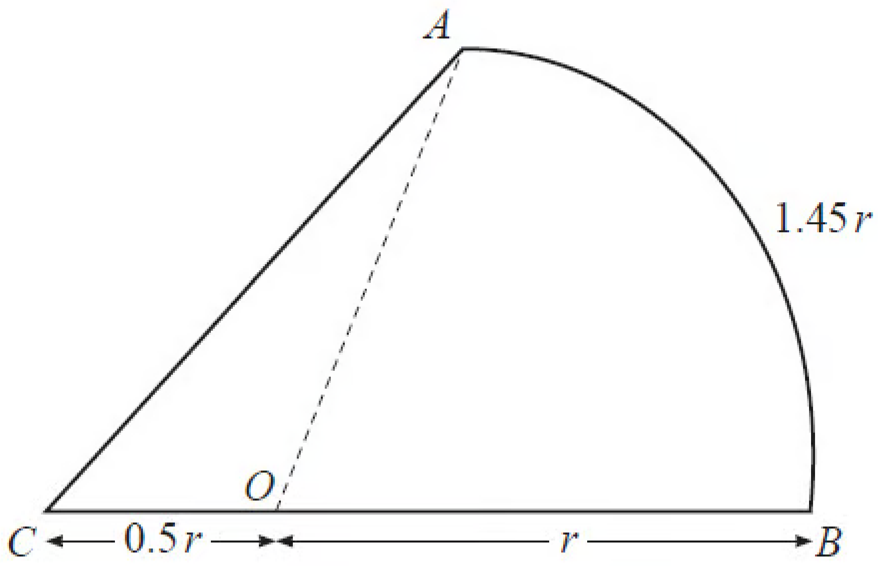
We need 2 of these so double your answer to get the total shaded area

$$2\left(3\pi - \frac{9\sqrt{3}}{4}\right)$$

$6\pi - \dfrac{9\sqrt{3}}{2}$ [1]

(4 marks)

3 (a) In this question all lengths are in centimetres.



The diagram shows the figure ABC. The arc AB is part of a circle, centre O, radius r, and is of length $1.45\ r$. The point O lies on the straight line CB such that $CO = 0.5\ r$.

Find, in radians, the angle AOB.

Answer

Arc length $= r \times \theta$

Therefore,

$$1.45r = r \times \theta$$
$$\theta = 1.45$$

Angle *AOB* = 1.45 radians [1]

(1 mark)

(b) Find the area of ABC, giving your in the form kr^2, where k is a constant.

Answer

Find the area of the sector.

$$\text{Area} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}(r^2)(1.45)$$

[1]

Find the area of triangle *COA* using $\frac{1}{2} ab \sin C$.

Angle *COA* sits on a straight line with Angle AOB. Therefore,

$$\text{Angle } COA = \pi - 1.45$$

Length *OA* is a radius of the circle, r. Length *CO* is $0.5r$.

Therefore,

$$\text{Area triangle } = \frac{1}{2}(r)(0.5r)(\sin(\pi - 1.45))$$

[1]

$$\text{Area triangle } = (0.25)(\sin(\pi - 1.45))r^2$$

Add together the area of the sector and the area of the triangle.

$$\text{Total area} = \frac{1}{2}(1.45)r^2 + (0.25)(\sin(\pi - 1.45))r^2$$

$$= 0.973178\ldots r^2$$

$0.973r^2$ [1]

(3 marks)

(c) Given that the perimeter of ABC is 12 cm, find the value of r.

Answer

Find the length *AC* using the Cosine rule.

$$AC^2 = (r)^2 + (0.5r)^2 - (2)(r)(0.5r)(\cos(\pi - 1.45))$$

[1]

$$AC^2 = r^2 + 0.25r^2 - r^2(-0.1205)$$

$$AC^2 = 1.3705r^2$$

$$AC = \sqrt{1.3705r^2}$$

$$AC = 1.17r$$

[1]

Add together the four lengths to find the perimeter of *ABC*.

$$0.5r + r + 1.45r + 1.17r = 4.12r$$

[1]

We are told that the perimeter of *ABC* is 12cm. Therefore,

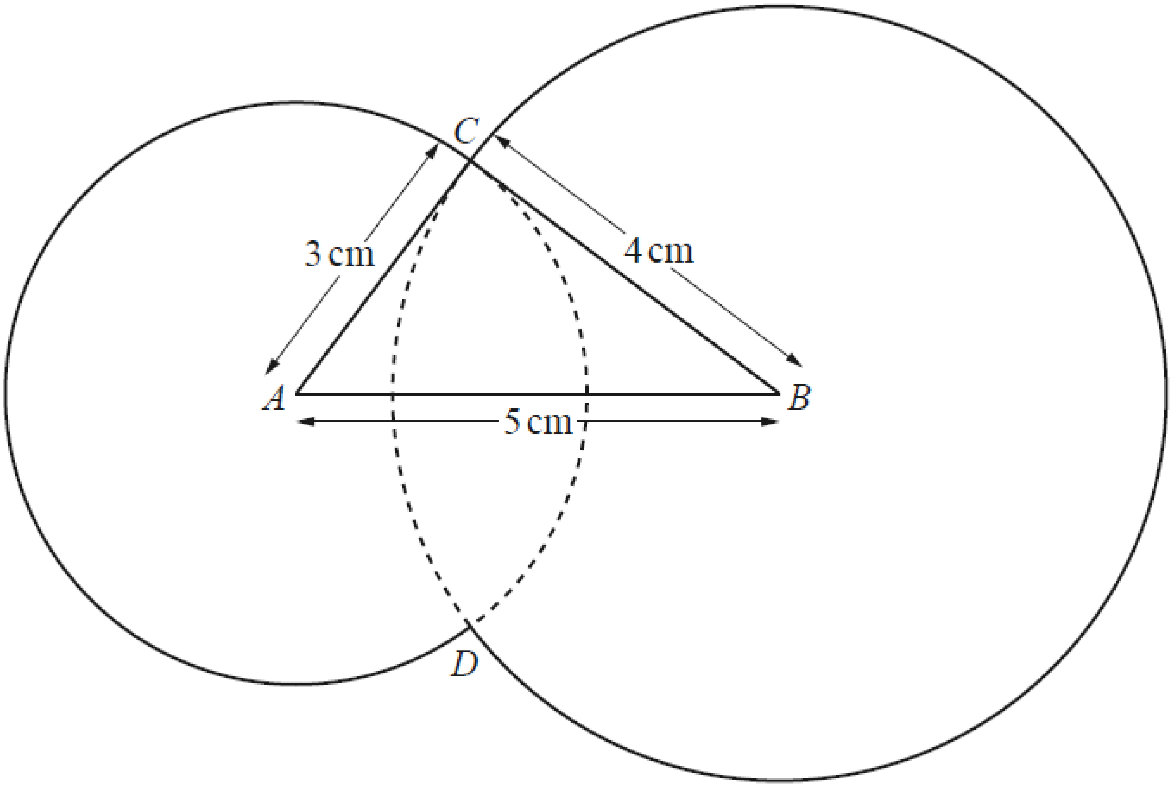
$$4.12r = 12$$

$$r = \frac{12}{4.12}$$

$r = 2.91$ cm [1]

(4 marks)

4 (a)



The diagram shows a shape consisting of two circles of radius 3 cm and 4 cm with centres A and B which are 5 cm apart. The circles intersect at C and D as shown. The lines AC and BC are tangents to the circles, centres B and A respectively. Find the angle CAB in radians.

Answer

Angle ACB is a right angle because CB is a tangent to the circle, and the angle between a tangent and a radius is a right-angle. It is also a Pythagorean triple 3, 4, 5 so we know it must be a right-angled triangle. Angle CAB can be found by

$$\tan\theta = \frac{4}{3}$$

[1]

Using the inverse tan function

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$\theta = 0.927$ [1]

(2 marks)

(b) Find the perimeter of the whole shape.

Answer

To find the perimeter, we need to use the formula for arc length, $l = r \times \theta$.

Angle ABC can be found by

$$\tan\theta = \frac{3}{4}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435...$$

[1]

The angle of the larger arc can be found by subtracting angle CBD from 2π

Angle CBD is twice angle ABC, so

$$\theta = 2\pi - 2 \times 0.644 = 4.995...$$

The length of the larger arc can be found by

$$4 \times 4.995 = 19.98...\text{cm}$$

[1]

The angle of the smaller arc can be found by

$$\theta = 2\pi - 2 \times 0.927 = 4.429...$$

The length of the smaller arc can be found by

$$3 \times 4.429 = 13.288...\text{cm}$$

[1]

The perimeter is the sum of the two lengths

33.3cm [1]

(4 marks)

(c) Find the area of the whole shape.

Answer

The area of a sector is given by $A = \frac{1}{2} r^2 \theta$.

The area of the larger sector is

$$A = \frac{1}{2} \times 4^2 \times (2\pi - 2 \times 0.644) = 39.96...\text{cm}^2$$

The area of the smaller sector is

$$A = \frac{1}{2} \times 3^2 \times (2\pi - 2 \times 0.927) = 19.93...\text{cm}^2$$

for either sector correct [1]

The area of the right-angled triangle is

$$A = \frac{1}{2} \times 4 \times 3 = 6\text{cm}^2$$

[1]

There are two right-angled triangles (the second being a reflection of the original in the line AB), so finding the sum of the two sectors and the two triangles gives

$$39.96 + 19.93 + 6 + 6$$

[1]

71.9cm^2 [1]

(4 marks)