IGCSE · Cambridge (CIE) · Further Maths

41 mins · 6 questions

Exam Questions

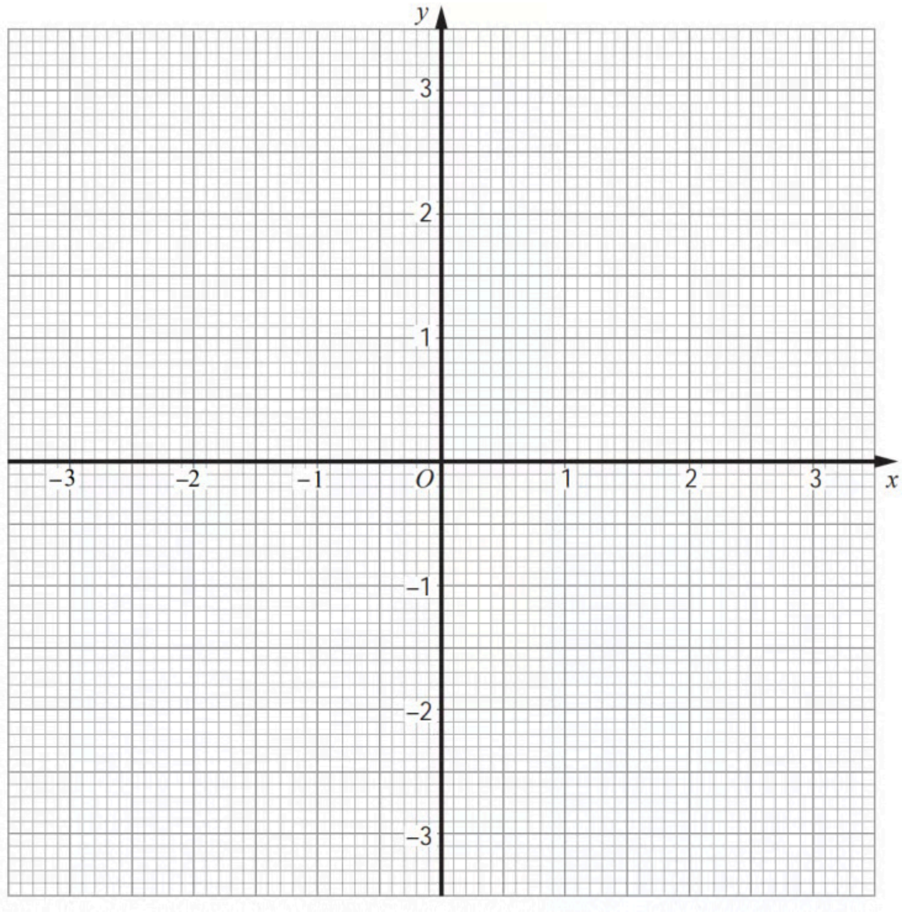
Coordinate Geometry of the Circle

Equation of a Circle / Tangents to Circles / Intersection of Two Circles

Medium (2 questions)	/10
Hard (3 questions)	/23
Very Hard (1 question)	/8
Total Marks	**/41**

Medium Questions

1 (a) Construct the graph of $x^2 + y^2 = 9$

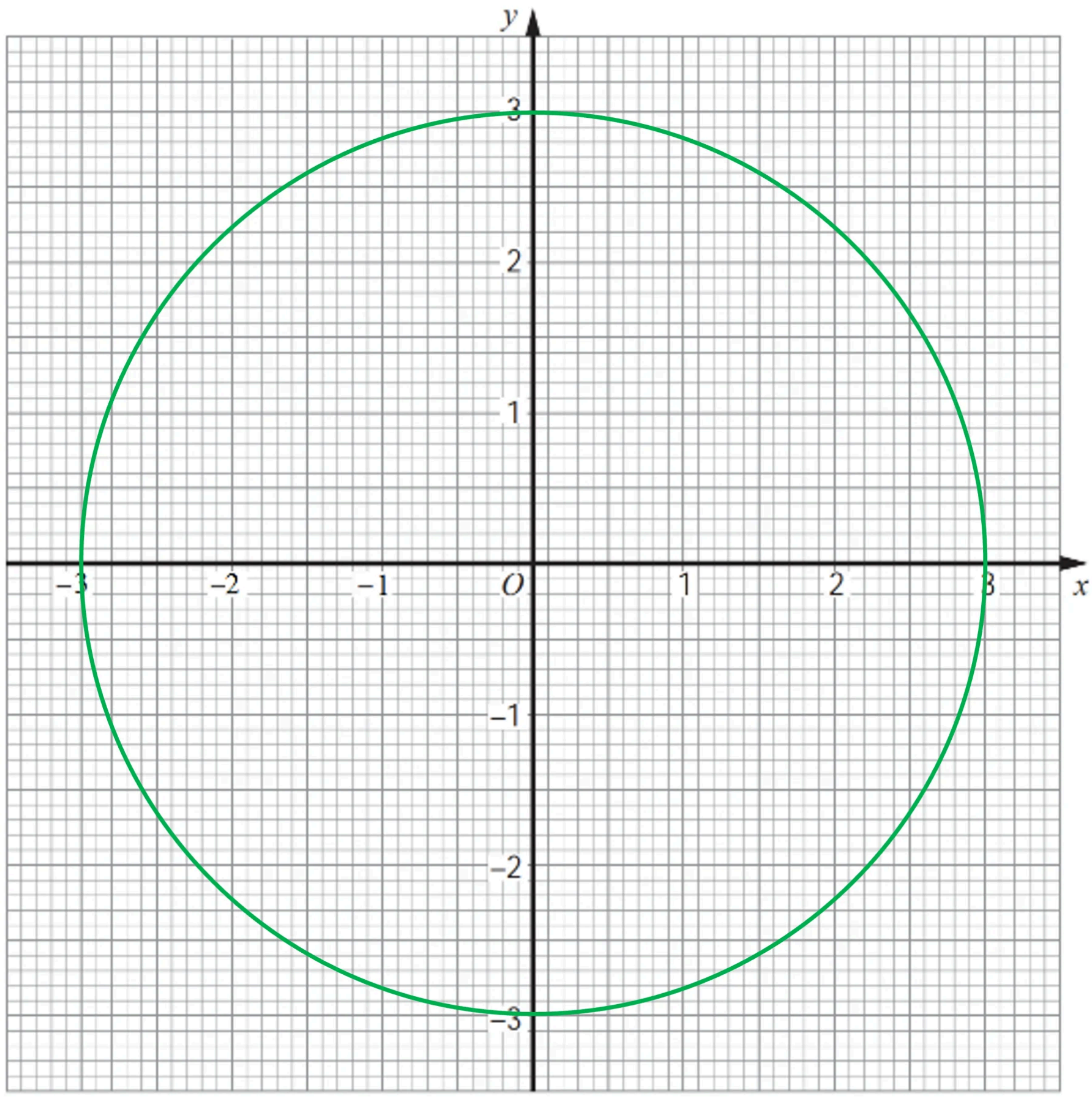


Answer

We should recognise the equation as the **equation of a circle** $x^2 + y^2 = r^2$ where r is the radius and the centre is (0, 0)
From this we can find the radius

$$r^2 = 9$$
$$r = 3$$

Now, using a pair of compasses, draw a circle with centre (0, 0) and radius 3. The circumference should pass through (3, 0), (0, 3), (−3, 0) and (0, −3)



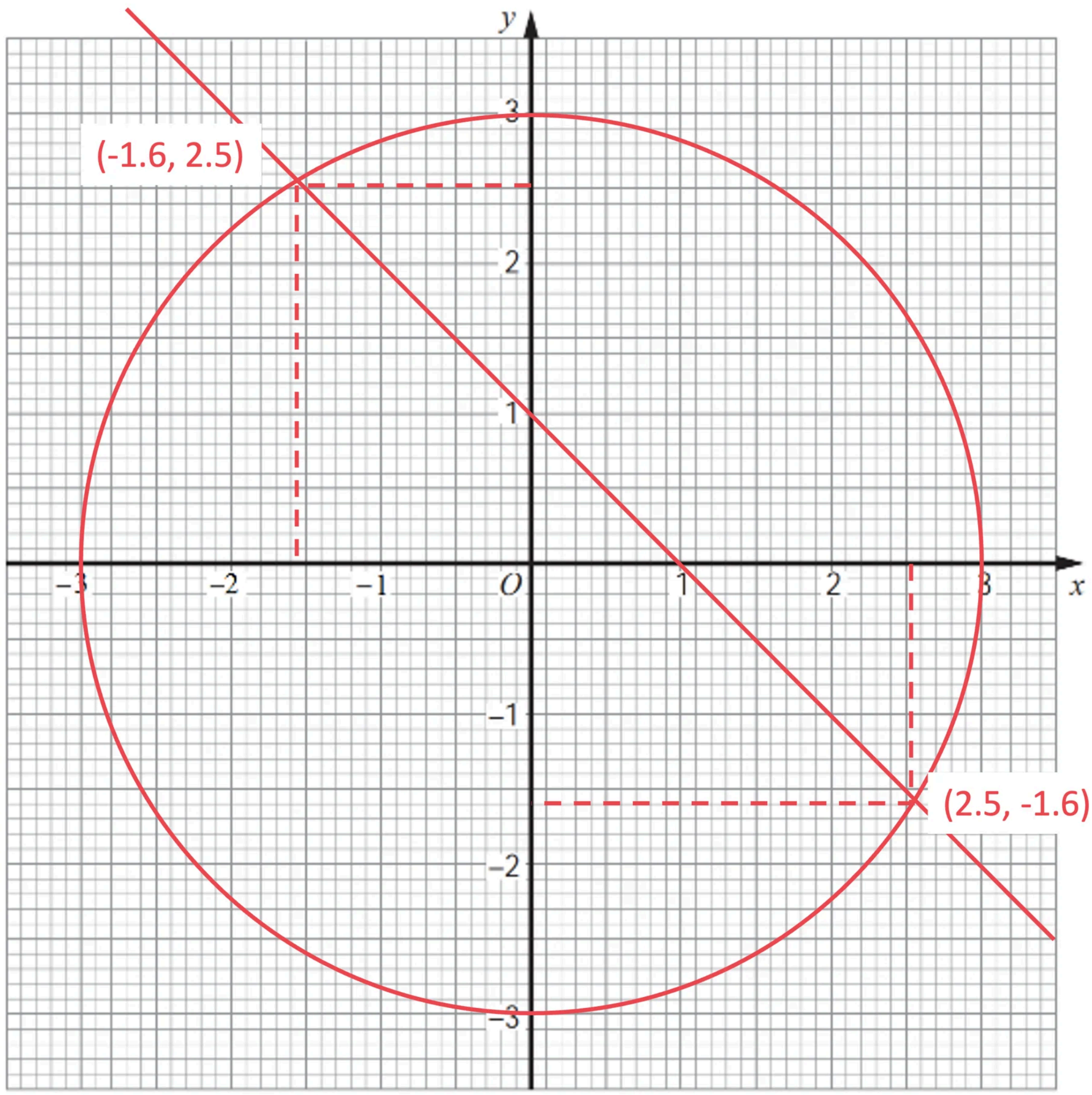
circle with centre (0, 0) [1]
circle with radius 3 [1]
(2 marks)

(b) By drawing the line $x + y = 1$ on the grid, solve the equations $x^2 + y^2 = 9$
$x + y = 1$

Answer

First we need to plot $x + y = 1$
Rearrange to $y = -x + 1$ and using your knowledge of **linear graphs** ***y* = *mx* + *c***, draw a

straight line through 1 on the y–axis with a gradient of -1

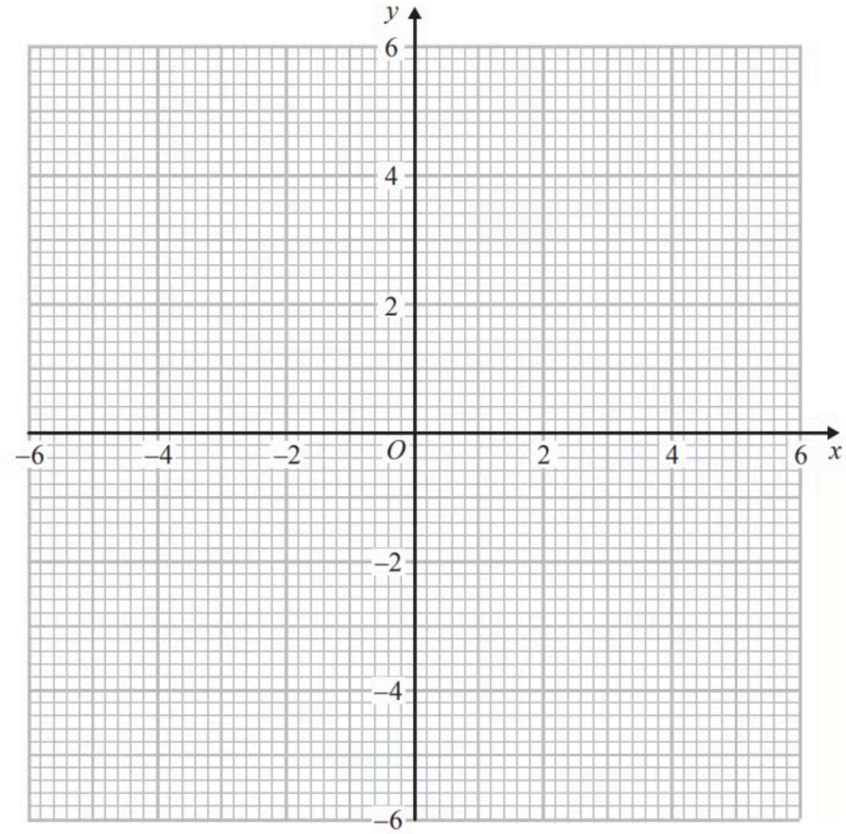


correct straight line [1]

Read the coordinates of where your straight line intersects with the circle drawn in part (a). Write these coordinates down as two pairs of x and y values There is some margin of error allowed, but your answers must be consistent with your graph

$x = 2.5$, $y = -1.6$ [1]
$x = -1.6$, $y = 2.5$ [1]
±0.2 is permitted for the above x and y values
(3 marks)

2 (a) On the grid, construct the graph of $x^2 + y^2 = 16$

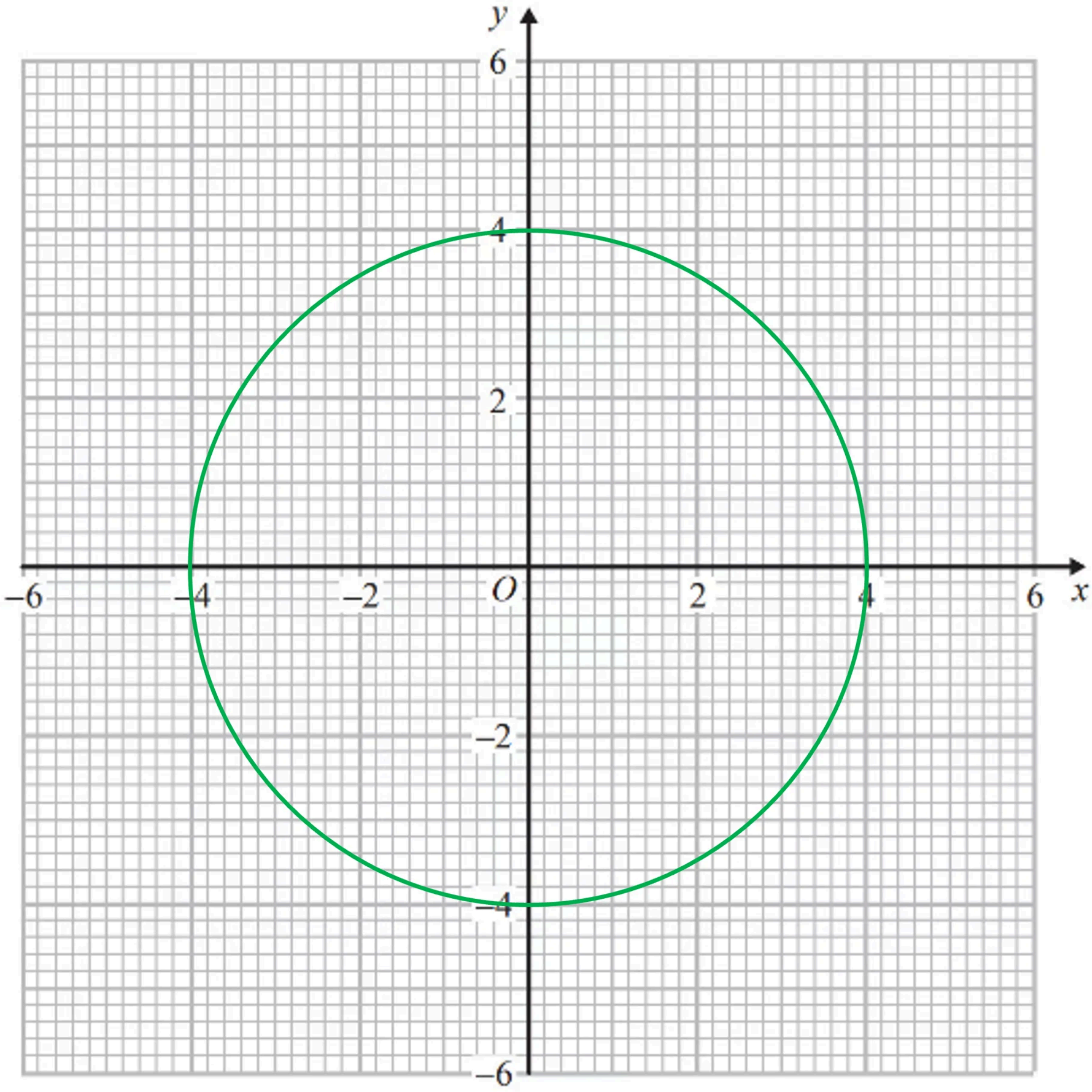


Answer

We should recognise the equation as the **equation of a circle** $x^2 + y^2 = r^2$ where r is the radius and the centre is (0, 0)
From this we can find the radius

$$r^2 = 16$$
$$r = 4$$

Now, using a pair of compasses, draw a circle with centre (0, 0) and radius 4. The circumference should pass through (4, 0), (0, 4), (−4, 0) and (0, −4)



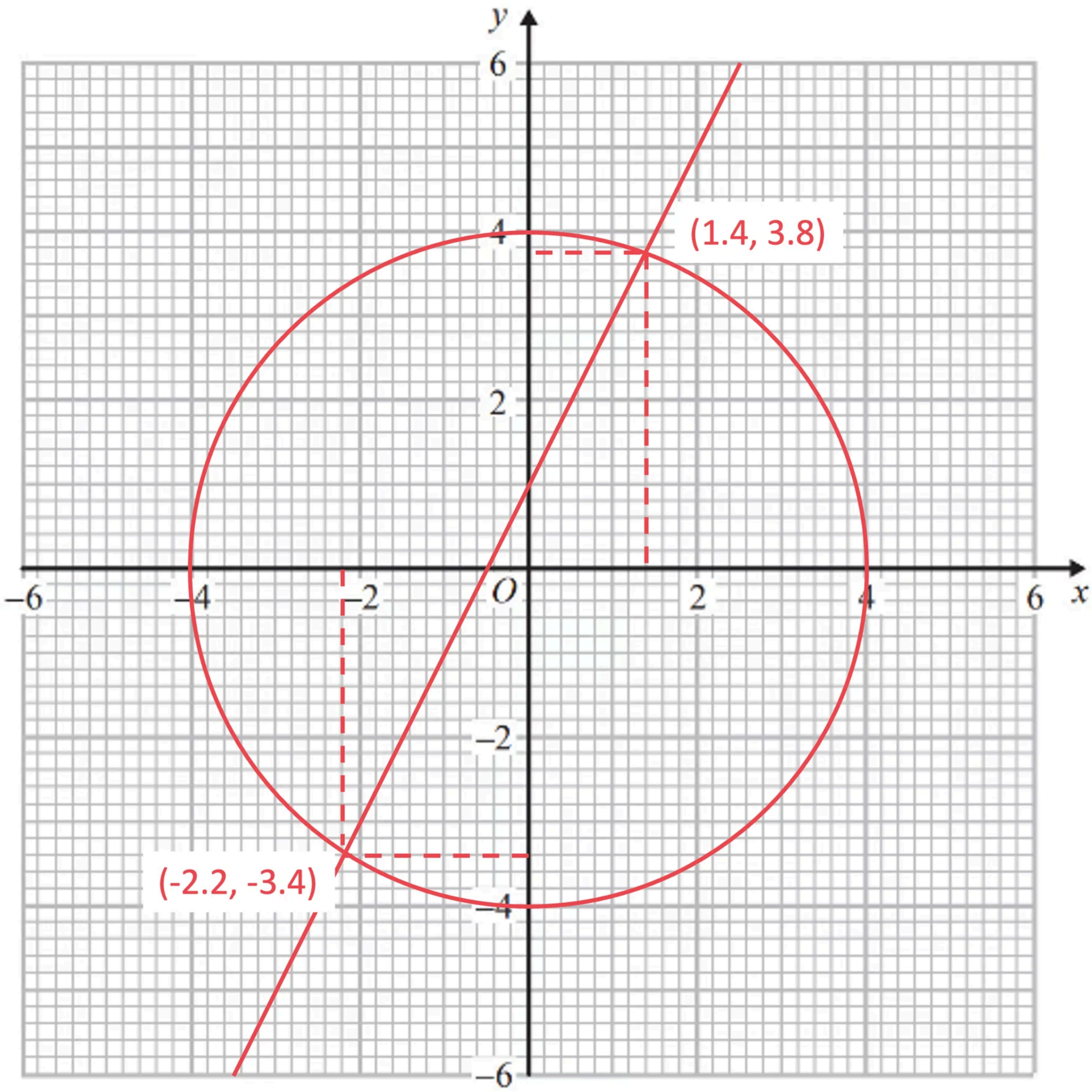
circle with centre (0, 0) [1]
circle with radius 4 [1]
(2 marks)

(b) Find estimates for the solutions of the simultaneous equations

$$x^2 + y^2 = 16$$
$$y = 2x + 1$$

Answer

First we need to plot $y = 2x + 1$. Using your knowledge of **linear graphs** ***y*** **=** ***mx*** **+** ***c***, draw a straight line through 1 on the *y*-axis with a gradient of 2



correct straight line [1]

Read the coordinates of where your straight line intersects with the circle drawn in part (a). Write these coordinates down as two pairs of *x* and *y* values

There is some margin of error allowed, but your answers must be consistent with your graph

$x = 1.4$, $y = 3.8$ [1]

$x = -2.2$, $y = -3.4$ [1]

±0.2 is permitted for the above x and y values

(3 marks)

Hard Questions

1 (a) **Solutions to this question by accurate drawing will not be accepted.** A circle has equation $x^2 + y^2 - 16x - 10y + 73 = 0$.

(i) Find the coordinates of the centre of the circle and the length of the radius.

(ii) Hence show that the point (10, 6.5) lies inside the circle.

Answer

i) Complete the square for x and for y in the given equation of the circle

$$x^2 - 16x + y^2 - 10y + 73 = 0$$
$$(x-8)^2 - 64 + (y-5)^2 - 25 + 73 = 0$$

[1]

$$(x-8)^2 + (y-5)^2 = 16$$

Remember that the equation of a circle is where (p, q) is the centre and r is the radius. Therefore the centre of the circle is

Centre: (8, 5) [1]

The radius is the square root of 16

r = 4 [1]

ii) Find the distance between (10, 6.5) and the centre of the circle, (8, 5) found in part (i)

$$\sqrt{(10-8)^2 + (6.5-5)^2}$$

[1]

$$= \sqrt{6.25}$$
$$= 2.5$$

This is less than the radius of the circle, which was identified as 4 in part (i)

2.5 < 4 (therefore (10, 6.5) is within the circle) [1]

(5 marks)

(b) A different circle has equation $(x - 10)^2 + (y - 6.5)^2 = 2.25$.

Show that the two circles touch. You are not required to find the coordinates of the common point.

Answer

The centre of this circle is (10, 6.5) and in part (a) we found that the distance from (10, 6.5) to the centre of the first circle is 2.5. Therefore *d*, the distance between centres, is 2.5

The radius of this new circle is

$$\sqrt{2.25} = 1.5$$

Two circles touch if $d = r_1 + r_2$ or $d = r_1 - r_2$

$4 - 1.5 = 2.5 \; (\therefore r_1 - r_2 = d)$ [1]

(1 mark)

2 (a) The circle C_1 has a centre at $(3, 4)$ and a radius of 3.

Write down the equation of C_1.

Answer

$(x-a)^2+(y-b)^2=r^2$ is the equation of a circle at centre (a, b) with radius r

$(x-3)^2+(y-4)^2=9$

[B1]

(1 mark)

(b) The circle C_2 has the equation

$$x^2+y^2-12x-16y+100-k^2=0$$

where $k>0$.

Given that C_2 is tangent to C_1, find the value of k in each of the following cases:

(i) The circles touch each other once and neither circle lies inside the other.

(ii) The circles touch each other once and C_1 lies inside C_2.

Answer

First find the centre and radius of C_2 by completing the square

$$x^2-12x+y^2-16y=k^2-100$$
$$(x-6)^2-36+(y-8)^2-64=k^2-100$$
$$(x-6)^2+(y-8)^2=k^2$$

[M1 A1]

This has centre $(6, 8)$ and radius k

Calculate d, the distance between centres of C_1 and C_2, $(3, 4)$ and $(6, 8)$

$$d=\sqrt{(6-3)^2+(8-4)^2}$$

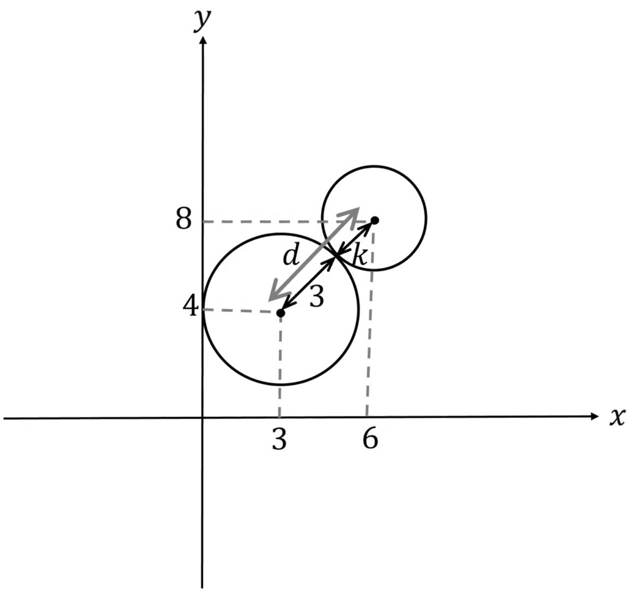
Simplify to find d

$$d = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

[B1]

(i)

It helps to sketch the two circles touching side-by-side (not one inside the other)



The sum of the two radii must equal d

$$d = 3 + k$$

[M1]

Mark Scheme and Guidance

This mark is for identifying the correct arrangement of the touching circles and attempting to equate the correct lengths.

Substitute in $d = 5$ and solve for k

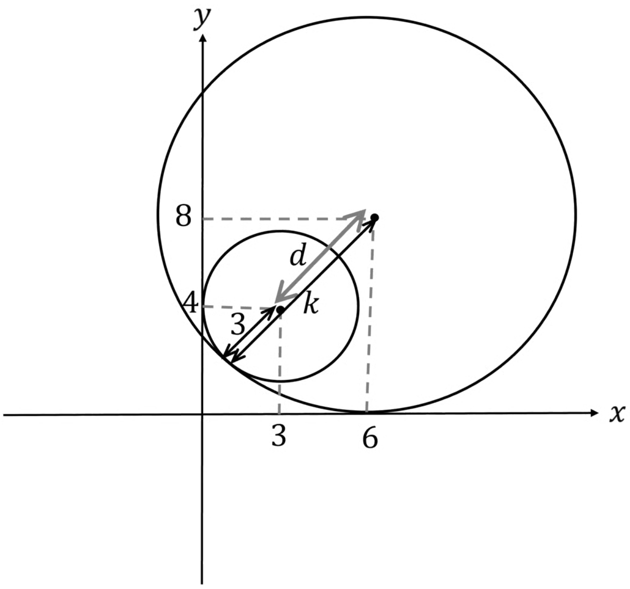
$$5 = 3 + k$$

$k = 2$

[A1]

(ii)

Draw another sketch of the circles touching but this time with C_1 inside C_2



The sum of the smaller radius and d must equal the bigger radius

$$3 + d = k$$

[M1]

Mark Scheme and Guidance

This mark is for identifying the correct arrangement of the touching circles and attempting to equate correct lengths.

Substitute in $d = 5$ and solve for k

$$3 + 5 = k$$

$k = 8$

[A1]
(7 marks)

(c) A different circle, C_3, intersects C_1 at two distinct points, one of which has the coordinates $(w, 7)$.

The equation of the common chord to C_1 and C_3 is $x + y = 10$.

Find w.

Answer

The equation of the common chord is the equation of the line connecting the two points of intersection

The quickest method to find w is to say $(w, 7)$ lies on $x + y = 10$

$$w + 7 = 10$$

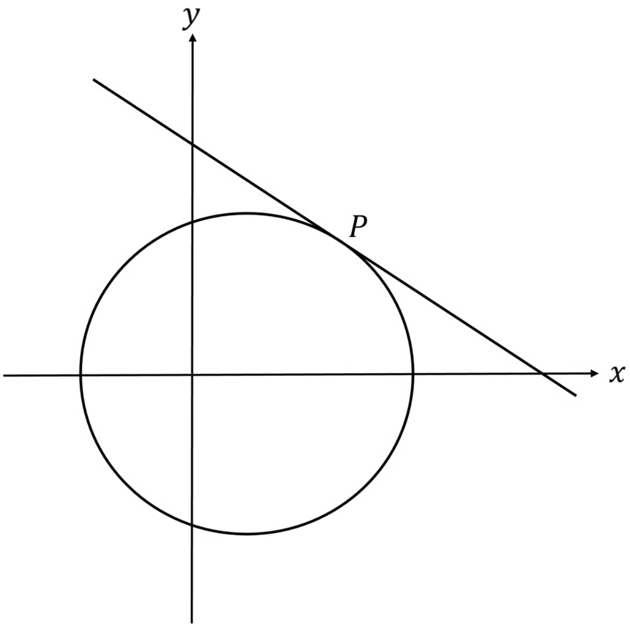
Then solve for w

$w = 3$

[B1]
(1 mark)

3 (a) The point $P(8, 8)$ lies on the circle $(x-2)^2 + y^2 = 100$ with centre C.

The circle and the tangent to the circle at P are shown below.



Find the equation of the tangent to the circle at P, giving your answer in the form $y = mx + c$.

Answer

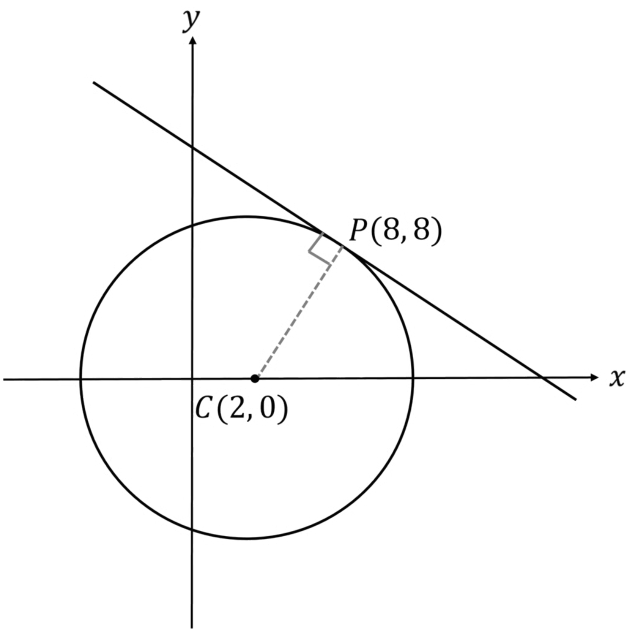
The tangent at P is perpendicular to the radius CP

Find the coordinates of C from $(x-a)^2 + (y-b)^2 = r^2$ having centre (a, b)

$$C(2, 0)$$

[B1]

The gradient of the tangent is perpendicular to the gradient of the radius CP



Find the gradient of the radius CP

$$m_{CP} = \frac{8-0}{8-2}$$
$$= \frac{4}{3}$$

Find the gradient perpendicular to the radius CP (i.e. the negative reciprocal)

$$m = -\frac{3}{4}$$

[M1]

Examiner Tips and Tricks

A quick check of the diagram confirms to you that this gradient is meant to be negative!

Substitute $(8,\ 8)$ and $m = -\frac{3}{4}$ into $y - y_1 = m(x - x_1)$ for the equation of the tangent

$$y-8=-\frac{3}{4}(x-8)$$

Rearrange into the form $y=mx+c$

$$y-8=-\frac{3}{4}x+6$$

$y=-\frac{3}{4}x+14$

[A1 A1]

Mark Scheme and Guidance

B1: For the correct coordinates of the centre of the circle.

M1: For attempting to find the perpendicular gradient of your radius gradient.

A1: For $y=mx+c$ with the correct gradient.

A1: For $y=mx+c$ with the correct y-intercept.

(4 marks)

(b) The tangent intersects the y-axis at the point Q.

Find the area of triangle PCQ.

Answer

The point Q has coordinates $(0,\ 14)$ from the answer in part (a)

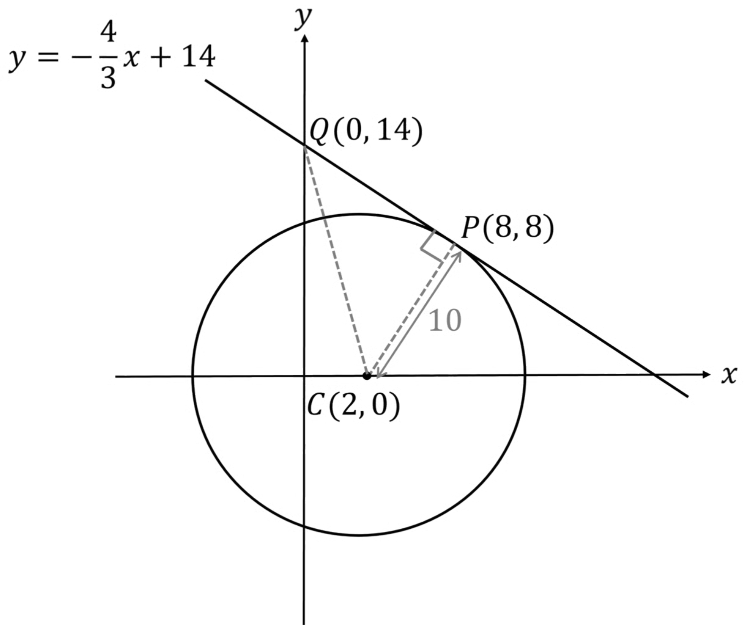
$$Q(0,\ 14)$$

Find the length of the radius CP from the equation $(x-2)^2+y^2=100=10^2$

$$CP=10$$

[B1]

Sketch triangle PCQ on the diagram and include all the information above



Triangle PCQ is right-angled so find its area using $\frac{1}{2} \times \text{base} \times \text{height}$ (where CP is the base and PQ is the height)

To find PQ use the length formula (or Pythagoras on the horizontal and vertical differences)

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(8-0)^2 + (8-14)^2}$$

[M1]

Mark Scheme and Guidance

This mark is attempting to find the length PQ using your Q coordinates.

Simplify

$$PQ = \sqrt{100}$$
$$= 10$$

[A1]

Use $\frac{1}{2} \times \text{base} \times \text{height}$ to find the area of triangle PCQ

$$\frac{1}{2} \times 10 \times 10$$

50

[A1]

(4 marks)

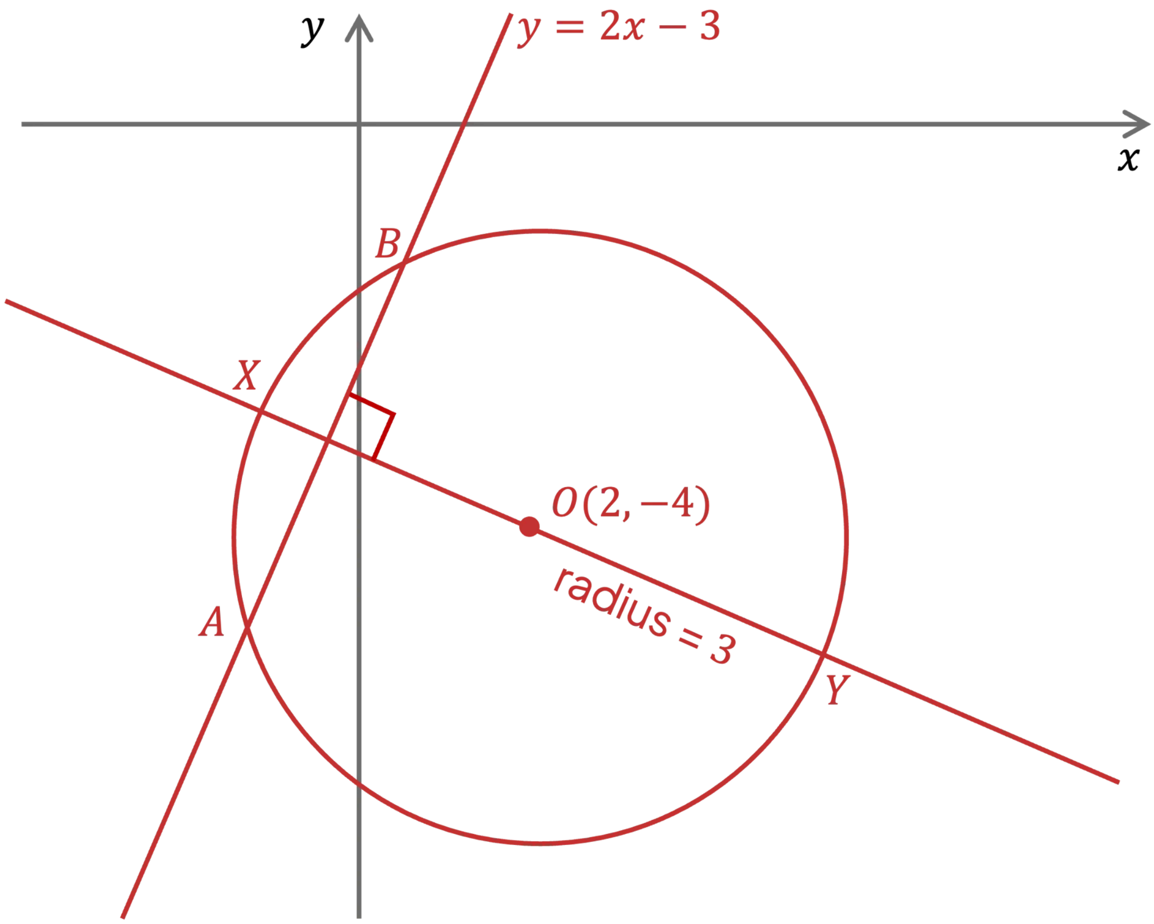
Very Hard Questions

1 A circle has a centre $(2, -4)$ and radius 3.
The line $y = 2x - 3$ intersects the circle at points A and B. The perpendicular bisector of line AB intersects the circle at points X and Y.

Find the area of kite $AXBY$.

Answer

It helps to visualise the problem if you sketch the information given



First write the equation of the circle using the information given.
$(x-h)^2 + (y-k)^2 = r^2$ where (h, k) is the centre and r is the radius

$$(x-2)^2 + (y+4)^2 = 3^2$$

[1]

Now substitute $y = 2x - 3$ into the equation of the circle to find the two possible x values of the intersection of the line and circle

$$(x-2)^2 + (2x-3+4)^2 = 3^2$$

[1]

Solve for x

$$\begin{aligned} x^2 - 4x + 4 + (2x+1)^2 &= 9 \\ x^2 - 4x + 4 + 4x^2 + 4x + 1 &= 9 \\ 5x^2 + 5 &= 9 \\ 5x^2 &= 4 \\ x^2 &= \frac{4}{5} \\ x &= \pm\frac{2}{\sqrt{5}} \end{aligned}$$

Substitute both possible values into the equation of the straight line in order to find pairs of coordinate points for *A* and *B*

$$y = 2\left(\frac{2}{\sqrt{5}}\right) - 3 \qquad y = 2\left(-\frac{2}{\sqrt{5}}\right) - 3$$

$$= \frac{4}{\sqrt{5}} - 3 \qquad = \frac{-4}{\sqrt{5}} - 3$$

$$A = \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} - 3\right) \qquad B = \left(\frac{-2}{\sqrt{5}}, \frac{-4}{\sqrt{5}} - 3\right)$$

one mark for each correct pair of coordinates [2]

(Note that *A* and *B* here correspond to the diagram we drew at the beginning of the question but are of course interchangeable depending on your diagram)

Find *AB* using Pythagoras' theorem

$$AB = \sqrt{\left(\frac{2}{\sqrt{5}} - \frac{-2}{\sqrt{5}}\right)^2 + \left(\frac{4}{\sqrt{5}} - 3 - \left(\frac{-4}{\sqrt{5}} - 3\right)\right)^2}$$

[1]

$$= \sqrt{\left(\frac{4}{\sqrt{5}}\right)^2 + \left(\frac{8}{\sqrt{5}}\right)^2}$$
$$= \sqrt{\frac{16}{5} + \frac{64}{5}}$$
$$= \sqrt{16}$$
$$= 4$$

[1]

XY is a perpendicular bisector of *AB* which is a chord. Therefore *XY* passes through the centre and therefore it is the diameter of the circle, so no real working is required! The radius is 3 so

$$XY = 6$$

[1]

The area of the kite is

$$\text{Area } = \frac{1}{2} \times 4 \times 6$$

12 [1]

(8 marks)