IGCSE · Cambridge (CIE) · Further Maths

4 hours · 32 questions

Exam Questions

Differentiation

Introduction to Differentiation / Differentiating Special Functions / Chain Rule / Product Rule / Quotient Rule / Applications of Differentiation / Second Order Derivatives / Modelling with Differentiation / Connected Rates of Change

Medium (12 questions)	/77
Hard (14 questions)	/102
Very Hard (6 questions)	/45
Total Marks	**/224**

Medium Questions

1 Given that $y = \tan x$, use calculus to find the approximate change in y as x increases from $-\frac{\pi}{4}$ to $h - \frac{\pi}{4}$, where h is small.

Answer

First note that $h - \frac{\pi}{4}$ is the same as $-\frac{\pi}{4} + h$, so this is a small increment in x.

Differentiate $y = \tan x$.

$$\frac{dy}{dx} = \sec^2 x$$

[1]

Substituting $x = -\frac{\pi}{4}$ into $\frac{dy}{dx}$ gives the gradient at of f at $x = -\frac{\pi}{4}$.

$$x = -\frac{\pi}{4}, \quad \frac{dy}{dx} = \sec^2\left(-\frac{\pi}{4}\right)$$

[1]

Use your calculator to evaluate – make sure your calculator is in radians though! Enter the "$\sec^2 x$" into your calculator.

$$\left(\frac{1}{\cos\left(-\frac{\pi}{4}\right)}\right)^2 = 2$$

As h is small we can use this gradient to estimate y using "change in y = gradient × change in x".

$$2 \times h = 2h$$

Approximate change in y is $2h$ [1]

(3 marks)

2 Find the x-coordinate of the stationary point on the curve $y = (2 - \sqrt{3})x^2 + x - 1$, giving your answer in the form $a + b\sqrt{3}$, where a and b are rational numbers.

Answer

Differentiate y with respect to x.

$$\frac{dy}{dx} = 2(2 - \sqrt{3})x + 1$$

[1]

Stationary points are where $\frac{dy}{dx} = 0$.

$$2(2 - \sqrt{3})x + 1 = 0$$

Rearrange to find x.

$$2(2 - \sqrt{3})x = -1$$

$$x = -\frac{1}{2(2 - \sqrt{3})}$$

Rationalise the denominator.

$$x = -\frac{1}{2(2 - \sqrt{3})} \times \frac{(2 + \sqrt{3})}{(2 + \sqrt{3})}$$

$$x = -\frac{2 + \sqrt{3}}{2(2 - \sqrt{3})(2 + \sqrt{3})}$$

$$x = -\frac{2 + \sqrt{3}}{2(4 - 3)}$$

$$x = -\frac{2 + \sqrt{3}}{2}$$

[1]

Simplify.

$$x = -1 - \frac{\sqrt{3}}{2}$$ [1]

(3 marks)

3 The radius, r cm, of a circle is increasing at the rate of 5 cms^{-1}. Find, in terms of π, the rate at which the area of the circle is increasing when $r = 3$.

Answer

This is a connected rates of change question – the rate of change of the radius is connected to the rate of change of the area.

The radius of the circle is increasing with respect to time.

$$\frac{dr}{dt} = 5$$

[1]

To find the rate at which the area is increasing, use chain rule.

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

The area of a circle is given by $A = \pi r^2$, so,

$$\frac{dA}{dr} = 2\pi r$$

[1]

Substitute the two results into chain rule.

$$\frac{dA}{dt} = 5 \times 2\pi r$$

$$\frac{dA}{dt} = 10\pi r$$

[1]

Substitute the given value of *r*.

$$\frac{dA}{dt} = 10\pi \times 3$$

$$\frac{dA}{dt} = 30\ \pi\ \text{cm}^2\ \text{s}^{-1}$$ [1]

(4 marks)

4 The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. The radius, r cm, of a sphere is increasing at the rate of $0.5\ \text{cms}^{-1}$. Find, in terms of π, the rate of change of the volume of the sphere when $r = 0.25$.

Answer

We know that the radius is increasing at a rate of 0.5cms^{-1}.

$$\frac{dr}{dt} = 0.5$$

[1]

Differentiate the volume of the sphere with respect to *r*.

$$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2$$

[1]

We want to find the rate of change of the volume which is $\frac{dV}{dt}$ so use the chain rule.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times 0.5$$

$$\frac{dV}{dt} = 2\pi r^2$$

[1]

Find $\frac{dV}{dt}$ when $r = \frac{1}{4}$.

$$\frac{dV}{dt} = 2\pi \times \left(\frac{1}{4}\right)^2 = \frac{\pi}{8}$$

$\frac{\pi}{8}$ cm^3 s^{-1} [1]

(4 marks)

5 (a) Given that $y = (x^2 - 1)\sqrt{5x+2}$, show that $\frac{dy}{dx} = \frac{Ax^2 + Bx + C}{2\sqrt{5x+2}}$, where A, B and C are integers.

Answer

Use the product rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and the chain rule.

Let $u = x^2 - 1$ and $v = (5x+2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \left((x^2-1) \times \frac{1}{2}(5x+2)^{-\frac{1}{2}} \times 5\right) + \left((5x+2)^{\frac{1}{2}} \times 2x\right)$$

for $\frac{5}{2}(5x+2)^{-\frac{1}{2}}$ [1]

for differentiation of a product [1]

fully correct differentiation [1]

Simplify each bracket.

$$\frac{dy}{dx} = \left(\frac{5}{2}(x^2-1)(5x+2)^{-\frac{1}{2}}\right) + \left(2x(5x+2)^{\frac{1}{2}}\right)$$

Take out common factor of $\frac{1}{2}(5x+2)^{-\frac{1}{2}}$.

$$\frac{dy}{dx} = \frac{1}{2}(5x+2)^{-\frac{1}{2}}(5(x^2-1)+4x(5x+2))$$

[1]

Expand and simplify, writing as a fraction.

$$\frac{dy}{dx} = \frac{5x^2-5+20x^2+8x}{2\sqrt{5x+2}}$$

Simplify.

$$\frac{dy}{dx} = \frac{25x^2+8x-5}{2\sqrt{5x+2}}$$ [1]

(5 marks)

(b) Find the coordinates of the stationary point of the curve $y=(x^2-1)\sqrt{5x+2}$ for $x>0$.
Give each coordinate correct to 2 significant figures.

Answer

From part (a), $\frac{dy}{dx} = \frac{25x^2+8x-5}{2\sqrt{5x+2}}$.

Stationary points occur when $\frac{dy}{dx}=0$.

$$\frac{25x^2+8x-5}{2\sqrt{5x+2}} = 0$$

$$25x^2+8x-5=0$$

[1]

Use the quadratic formula (and/or calculator) to solve the quadratic equation.

$$x = \frac{-8 \pm \sqrt{8^2 - 4(25)(-5)}}{2(25)}$$

$$x = 0.3149... \text{ or } -0.6349...$$

$$x > 0 \text{ so } x = 0.3149...$$

Find y by substituting x into the equation of the curve.

$$y = (x^2 - 1)\sqrt{5x+2}$$
$$y = (0.3149...^2 - 1)\sqrt{5(0.3149...)+2}$$
$$y = -1.7031...$$

$x = 0.315$ (3 s.f.) [1]

$y = -1.70$ (3 s.f) [1]

(3 marks)

(c) Determine the nature of this stationary point.

Answer

Consider the gradient of the curve either side of the stationary point

From part (a), we know that $\frac{dy}{dx} = \frac{25x^2 + 8x - 5}{2\sqrt{5x+2}}$

The stationary point occurs at $(0.315, -1.70)$ so look at the gradient either side of this point.

When $x = 0.3$,

$$\frac{dy}{dx} = \frac{25(0.3)^2 + 8(0.3) - 5}{2\sqrt{5(0.3)+2}}$$
$$\frac{dy}{dx} = -0.093... < 0$$

When $x = 0.4$,

$$\frac{dy}{dx} = \frac{25(0.4)^2 + 8(0.4) - 5}{2\sqrt{5(0.4) + 2}}$$

$$\frac{dy}{dx} = 0.55 > 0$$

[1]

The gradient changes from positive to negative through the point $(0.315, -1.70)$ so the curve is "U -shaped".

$\therefore$ $(0.315, -1.70)$ **is a minimum point** [1]

(2 marks)

6 Variables x and y are such that $y = \sin x + e^{-x}$. Use differentiation to find the approximate change in y as x increases from $\frac{\pi}{4}$ to $\frac{\pi}{4} + h$, where h is small.

Answer

Differentiate y with respect to x.

$$y = \sin x + e^{-x}$$

$$\frac{dy}{dx} = \cos x - e^{-x}$$

at least one term correct [1]
all correct [1]

Substitute $x = \frac{\pi}{4}$ into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \cos\left(\frac{\pi}{4}\right) - e^{-\frac{\pi}{4}}$$

Evaluate, ensuring your calculator is in radian mode.

$$\frac{dy}{dx} = 0.2511...$$

[1]

As h is small, use "change in $y = gradient \times$ change in x".

$$\text{change in y} = 0.251\ldots \times h$$

The approximate change in y is $0.25h$ [1]

(4 marks)

7 (a) The equation of a curve is $y = x\sqrt{16 - x^2}$ for $0 \le x \le 4$.

Find the exact coordinates of the stationary point of the curve.

Answer

Stationary points occur when $\frac{dy}{dx} = 0$ so first differentiate using the product rule, for $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

$$u = x \qquad v = (16 - x^2)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \frac{1}{2}(16 - x^2)^{-\frac{1}{2}} \times -2x = -x(16 - x^2)^{-\frac{1}{2}}$$

[1]

Apply the product rule.

$$\frac{dy}{dx} = (16 - x^2)^{\frac{1}{2}} \times 1 + x \times \left(-x(16 - x^2)^{-\frac{1}{2}}\right)$$

use product rule [1]
fully correct [1]

Simplify and set equal to 0.

$$0 = (16 - x^2)^{\frac{1}{2}} - x^2(16 - x^2)^{-\frac{1}{2}}$$

[1]

Adding $x^2(16 - x^2)^{-\frac{1}{2}}$ to both sides gives.

$$x^2(16 - x^2)^{-\frac{1}{2}} = (16 - x^2)^{\frac{1}{2}}$$

$$\frac{x^2}{(16 - x^2)^{\frac{1}{2}}} = (16 - x^2)^{\frac{1}{2}}$$

Multiplying both sides by $(16 - x^2)^{\frac{1}{2}}$.

$$x^2 = 16 - x^2$$

Adding x^2 to both sides.

$$2x^2 = 16$$

$$x^2 = 8$$

[1]

Finding the square root of both sides - disregard the negative root as it's outside the range given in the question.

$$x = \sqrt{8} = 2\sqrt{2}$$

Substituting this into the equation gives the y co-ordinate.

$$y = 8$$

$(2\sqrt{2}, 8)$ [1]

(6 marks)

(b) Find $\frac{d}{dx}(16 - x^2)^{\frac{3}{2}}$ and hence evaluate the area enclosed by the curve $y = x\sqrt{16 - x^2}$ and the lines $y = 0$, $x = 1$ and $x = 3$.

Answer

Differentiating using the chain rule that states that $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

$$y = u^{\frac{3}{2}} \qquad u = 16 - x^2$$

$$\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}} \qquad \frac{du}{dx} = -2x$$

Substituting into the chain rule gives

$$\frac{dy}{dx} = -2x \times \frac{3}{2}u^{\frac{1}{2}}$$

attempt at chain rule [1]

Substituting in the expression for u

$$\frac{dy}{dx} = -2x \times \frac{3}{2}(16 - x^2)^{\frac{1}{2}}$$

[1]

Simplifying gives

$$\frac{dy}{dx} = -3x(16 - x^2)^{\frac{1}{2}}$$

The area required is

$$\int_1^3 x(16 - x^2)^{\frac{1}{2}}\,dx$$

You have just differentiated $(16 - x^2)^{\frac{3}{2}}$ to get $-3x(16 - x^2)^{\frac{1}{2}}$, which is almost what you are being asked to integrate to find the area. The area is minus-a-third of the result above, so you can write down

$$\int_1^3 x(16 - x^2)^{\frac{1}{2}} \, dx = \left[-\frac{1}{3}(16 - x^2)^{\frac{3}{2}}\right]_1^3$$

for $-\frac{1}{3}$ [1]

for $k(16 - x^2)^{\frac{3}{2}}$ [1]

Applying the limits

$$\left(-\frac{1}{3}(16 - 3^2)^{\frac{3}{2}}\right) - \left(-\frac{1}{3}(16 - 1^2)^{\frac{3}{2}}\right)$$

13.2 [1]

(5 marks)

8 (a) A curve has equation $y = (2x-1)\sqrt{4x+3}$.

Show that $\frac{dy}{dx} = \frac{4(Ax+B)}{\sqrt{4x+3}}$, where A and B are constants.

Answer

Differentiate $y=(2x-1)\sqrt{4x+3}$ using the product rule and chain rule.

$$u=2x-1 \qquad v=(4x+3)^{\frac{1}{2}}$$

$$\frac{du}{dx}=2 \qquad \frac{dv}{dx}=2(4x+3)^{-\frac{1}{2}}$$

$\frac{dv}{dx}=2(4x+3)^{-\frac{1}{2}}$ [1]

$$\frac{dy}{dx}=(2)(4x+3)^{\frac{1}{2}}+(2x-1)\left(2(4x+3)^{-\frac{1}{2}}\right)$$

use of product rule [1]
all terms correct [1]

Simplify – start by taking factors of 2 and $(4x+3)^{-\frac{1}{2}}$.

$$\frac{dy}{dx}=2(4x+3)^{-\frac{1}{2}}[4x+3+2x-1]$$

[1]

$$\frac{dy}{dx}=2(4x+3)^{-\frac{1}{2}}[6x+2]$$

Rewrite as a fraction.

$$\frac{dy}{dx}=\left(\frac{2}{(4x+3)^{\frac{1}{2}}}\right)(6x+2)$$

Multiply the fractions.

$$\frac{dy}{dx} = \frac{12x + 4}{(4x + 3)^{\frac{1}{2}}}$$

Factorise the numerator, and rewrite the denominator to express as required.

$$\frac{dy}{dx} = \frac{4(3x + 1)}{\sqrt{4x + 3}} \text{ [1]}$$

(5 marks)

(b) Hence write down the x-coordinate of the stationary point of the curve.

Answer

At a stationary point, the gradient is 0, and therefore $\frac{dy}{dx} = 0$.

$$\frac{4(3x + 1)}{\sqrt{4x + 3}} = 0$$

Rearrange and solve for x.

$$4(3x + 1) = 0$$
$$3x + 1 = 0$$
$$x = -\frac{1}{3}$$

$$x = -\frac{1}{3} \text{ [1]}$$

(1 mark)

(c) Determine the nature of this stationary point.

Answer

To determine the nature of the stationary point, we can find the gradient *just* either side of $x = -\frac{1}{3}$. When $x = -\frac{1}{2}$ (to the left of the stationary point),

$$\frac{dy}{dx} = \frac{4\left(3\left(-\frac{1}{2}\right)+1\right)}{\sqrt{4\left(-\frac{1}{2}\right)+3}}$$

$$\frac{dy}{dx} = -2$$

When $x = -\frac{1}{4}$ (to the right of the stationary point),

$$\frac{dy}{dx} = \frac{4\left(3\left(-\frac{1}{4}\right)+1\right)}{\sqrt{4\left(-\frac{1}{4}\right)+3}}$$

$$\frac{dy}{dx} = \frac{\sqrt{2}}{2}$$

[1]

The gradient goes from negative to positive through the point where $x = -\frac{1}{3}$, so the curve is "$\cup$-shaped".

$\therefore\ x = -\frac{1}{3}$ is a minimum point [1]

(2 marks)

9 (a) Find the equation of the tangent to the curve $y = x^3 - 6x^2 + 3x + 10$ at the point where $x = 1$.

Answer

Substitute $x = 1$ into the equation to find the y-coordinate.

$$y = 1^3 - 6(1)^2 + 3(1) + 10$$

$$y = 8$$

[1]

Differentiate the equation of the curve to find the gradient function.

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

[1]

Substitute $x = 1$ to find the gradient at this point.

$$\frac{dy}{dx} = 3(1)^2 - 12(1) + 3$$

$$\frac{dy}{dx} = -6$$

[1]

Find the equation of the tangent with point $(1, 8)$ and gradient -6.

$$y - 8 = -6(x - 1)$$

$$y - 8 = -6x + 6$$

$$y = -6x + 14$$

$y = -6x + 14$ [1]

(4 marks)

(b) Find the coordinates of the point where this tangent meets the curve again.

Answer

From part (a), the equation of the tangent is $y = -6x + 14$. Equate the tangent and the curve.

$$-6x + 14 = x^3 - 6x^2 + 3x + 10$$

Rearrange to get a cubic equation.

$$14 = x^3 - 6x^2 + 9x + 10$$
$$x^3 - 6x^2 + 9x - 4 = 0$$

[1]

From part (a) we know $x = 1$ is a solution.

$(x - 1)$ is a factor of the cubic

[1]

Factorise the cubic using long division (or inspection).

$$\begin{array}{r|rrrrl} & 1x^2 & -5x & +4 & & \\ \hline x-1 & 1x^3 & -6x^2 & +9x & -4 & \\ & 1x^3 & -1x^2 & +0x & +0 & - \\ \hline & & -5x^2 & +9x & -4 & \\ & & -5x^2 & +5x & +0 & - \\ \hline & & & 4x & -4 & \\ & & & 4x & -4 & - \\ \hline & & & & 0 & \end{array}$$

The factorised cubic is

$$(x - 1)(x^2 - 5x + 4) = 0$$

[1]

Factorise the second bracket.

$$(x - 1)(x - 1)(x - 4) = 0$$

[1]

$(x-1)$ is a repeated factor and is the solution we already know from part (a). The other solution will give us the x-coordinate of where the tangent meets the curve again.

$$x-4=0$$

$$x=4$$

Substitute $x=4$ into the equation of the tangent to find the y-coordinate.

$$y=-6(4)+14=-10$$

The tangent and curve meet again at the point $(4,\ -10)$ [1]

(5 marks)

10 (a) It is given that $y = \ln(1 + \sin x)$ for $0 < x < \pi$.

Find $\frac{dy}{dx}$

Answer

To differentiate $y = \ln(1 + \sin x)$, we can use the fact that the differential of $\ln x$ is $\frac{1}{x}$ and chain rule.

$$\frac{dy}{dx} = \frac{1}{1 + \sin x} \times \cos x$$

$\frac{1}{1 + \sin x}$ [1]

$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$ [1]

(2 marks)

(b) Find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$, giving your answer in the form $\frac{1}{\sqrt{a}}$, where a is an integer.

Answer

From part (*a*), we know that $\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$.

Substitute $x = \frac{\pi}{6}$ into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\cos\left(\frac{\pi}{6}\right)}{1 + \sin\left(\frac{\pi}{6}\right)}$$

[1]

$$\frac{dy}{dx} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{3}$$

Rewrite in the form as specified in the question.

$$\frac{dy}{dx} = \frac{1}{\sqrt{3}}$$ [1]

(2 marks)

(c) Find the values of x for which $\frac{dy}{dx} = \tan x$.

Answer

From part (a), we know that $\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$. Equate to $\tan x$ to form a trigonometric equation.

$$\frac{\cos x}{1 + \sin x} = \tan x$$

Use $\tan x = \frac{\sin x}{\cos x}$ to simplify.

$$\frac{\cos x}{1 + \sin x} = \frac{\sin x}{\cos x}$$

[1]

Rearrange.

$$\frac{\cos x}{1 + \sin x} \times \cos x = \sin x$$

$$\cos^2 x = (1 + \sin x)\sin x$$

$$\cos^2 x = \sin x + \sin^2 x$$

Use $\cos^2 x = 1 - \sin^2 x$ to make the equation in terms of $\sin x$ only.

$$1 - \sin^2 x = \sin x + \sin^2 x$$

Rearrange.

$$2\sin^2 x + \sin x - 1 = 0$$

[1]

Factorise this quadratic equation and solve.

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\sin x = \frac{1}{2}, \quad \sin x = -1$$

$\sin x = -1$ has no solutions in the range $0 < x < \pi$. $\sin x = \frac{1}{2}$ has two solutions in the range $0 < x < \pi$.

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Considering the graph of $y = \sin x$ or using $\sin(\pi - x) = \sin x$, the second solution can be found.

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$\boldsymbol{x = \frac{\pi}{6}}$ and $\boldsymbol{x = \frac{5\pi}{6}}$ [2]

1 mark for each

(5 marks)

11 (a) A curve has equation $y = x\cos x$.

Find $\dfrac{dy}{dx}$.

Answer

Differentiate using product rule, $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

$$u = x \quad v = \cos x$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = -\sin x$$

[1]

$$\frac{dy}{dx} = (x)(-\sin x) + (1)(\cos x)$$

$\dfrac{dy}{dx} = -x\sin x + \cos x$ [1]

(2 marks)

(b) Find the equation of the normal to the curve at the point where $x = \pi$, giving your answer in the form $y = mx + c$.

Answer

Find the gradient of the curve at the point $x = \pi$.

$$\frac{dy}{dx} = -\pi\sin\pi + \cos\pi$$

$$\frac{dy}{dx} = -1$$

[1]

The gradient of the normal is the negative reciprocal of this.

$$\therefore\ m = 1$$

[1]

Substitute $x = \pi$ into the equation of the curve to find the corresponding $y$value.

$$y = \pi \cos \pi$$
$$y = -\pi$$

The normal passes through the point

$$(\pi, -\pi)$$

[1]

We know the gradient and a point the normal passes through so use the point-gradient form for the equation of a line.

$$y - (-\pi) = 1(x - \pi)$$

Write in the form $y = mx + c$ as required for the final answer.

$y = x - 2\pi$ [1]

(4 marks)

12 Find the equation of the tangent to the curve $y = \dfrac{\ln(3x^2 - 1)}{x + 2}$ at the point where $x = 1$.

Give your answer in the form $y = mx + c$, where m and c are constants correct to 3 decimal places.

Answer

Use the quotient rule, i.e. for $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

$$u = \ln(3x^2 - 1),\ v = x + 2$$

Use the chain rule to differentiate the log function.

$$\frac{du}{dx} = \frac{6x}{3x^2 - 1}$$

[1]

$$\frac{dv}{dx} = 1$$

Apply the quotient rule.

$$\frac{dy}{dx} = \frac{(x+2)\frac{6x}{3x^2-1} - \ln(3x^2-1)}{(x+2)^2}$$

[1]

The gradient at the point $x = 1$ is required.

$$\frac{dy}{dx} = \frac{9 - \ln 2}{9} = 0.922\,983...$$

[1]

Substitute $x = 1$ to the original equation to find the y-coordinate.

$$y = 0.231\,049...$$

[1]

Substitute these values into $y = mx + c$ to find the equation of the tangent.

$$0.231\,049... = 0.922\,983... \times 1 + c$$
$$c = -0.691\,934...$$

[1]

Final answer requires values rounded to 3 decimal places.

$y = 0.923x - 0.692$ [1]

(6 marks)

Hard Questions

1 A curve has equation $y = \ln(5 - 3x)$ where $x < \frac{5}{3}$. The normal to the curve at the point where $x = -5$, cuts the x-axis, at the point P.
Find the equation of the normal and the x-coordinate of P.

Answer

Differentiate $y = \ln(5 - 3x)$.

$$\frac{dy}{dx} = \frac{1}{5-3x} \times (-3)$$
$$= \frac{-3}{5-3x}$$

$\frac{k}{5-3x}$ *where* $k \neq -3$ [1]

$\frac{-3}{5-3x}$ [1]

Find the gradient of the tangent to $y = \ln(5 - 3x)$ when $x = -5$ by substituting $x = -5$ into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-3}{5-3(-5)}$$
$$= -\frac{3}{20}$$

[1]

Divide this by −1 to find m, the gradient of the normal.

$$m = \frac{-1}{-\frac{3}{20}} = \frac{20}{3}$$

[1]

Find y when $x = -5$.

$$y = \ln(5 - 3(5)) = \ln 20$$

[1]

Substitute $(-5, \ln 20)$ and $m = \frac{20}{3}$ into the equation of a straight line. Here we will use the point–gradient form, $y - y_1 = m(x - x_1)$.

$$y - \ln 20 = \frac{20}{3}(x - (-5))$$

Therefore the equation of the normal is

$$\mathbf{y - \ln 20 = \frac{20}{3}(x + 5)}$$ [1]

Equivalent forms of the equation are acceptable

Substitute $y = 0$ to find where line crosses the x-axis.

$$0 - \ln 20 = \frac{20}{3}(x + 5)$$

$$-\ln 20 \div \frac{20}{3} = x + 5$$

$$\frac{-3\ln 20}{20} - 5 = x$$

$$x = -5.44935984...$$

$$\mathbf{x = -\frac{3\ln 20}{20} - 5 \quad \text{or} \quad -5.45 \ (3 \text{ s.f.})}$$ [1]

(7 marks)

2 Variables x and y are such that $y = e^{\frac{x}{2}} + x\cos 2x$, where x is in radians. Use differentiation to find the approximate change in y as x increases from 1 to $1 + h$, where h is small.

Answer

Differentiate $e^{\frac{x}{2}}$.

$$\frac{1}{2}e^{\frac{x}{2}}$$

[1]

Differentiate $x\cos 2x$ using the product rule.

$$u = x \text{ and } v = \cos 2x$$

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = -2\sin 2x$$

$$(x \times -2\sin 2x) + (1 \times \cos 2x)$$

for $-2\sin 2x$ [1]

$$= -2x\sin 2x + \cos 2x$$

[1]

Combine the two results for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}} - 2x\sin 2x + \cos 2x$$

[1]

Substitute $x = 1$.

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}} - 2\sin 2 + \cos 2$$

[1]

$$= -1.410381055$$

Change in $y = \frac{\mathrm{d}y}{\mathrm{d}x} \times$ change in x.

Change in $x = (1 + h) - (1) = h$.

$$\text{change in } y = -1.410381055 \times h$$

Approximate change in y is $-1.41h$ [1]

(6 marks)

3 The tangent to the curve $y = \ln(3x^2 - 4) - \frac{x^3}{6}$, at the point where $x = 2$, meets the y-axis at the point P. Find the exact coordinates of P.

Answer

We need to find the equation of the tangent to the curve so start by differentiating the curve to find the gradient of the tangent.
Use chain rule to differentiate $\ln(3x^2 - 4)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x}{3x^2 - 4} - \frac{3x^2}{6}$$

$\frac{6x}{3x^2 - 4}$ [1]

Simplify.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x}{3x^2 - 4} - \frac{x^2}{2}$$

[1]

Find the gradient when $x = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6(2)}{3(2)^2 - 4} - \frac{(2)^2}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$$

[1]

To find the equation of the tangent, we also need a point and the gradient. So find y when $x = 2$.

$$y = \ln(3(2)^2 - 4) - \frac{(2)^3}{6}$$

$$y = \ln(8) - \frac{4}{3}$$

[1]

Find the equation of the tangent using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(2, \ln(8) - \frac{4}{3}\right)$ and $m = -\frac{1}{2}$.

$$y - \left(\ln(8) - \frac{4}{3}\right) = -\frac{1}{2}(x - 2)$$

[1]

At P, $x = 0$ because it is the y-axis so substitute $x = 0$ into the equation of the tangent.

$$y - \left(\ln(8) - \frac{4}{3}\right) = -\frac{1}{2}(0 - 2)$$

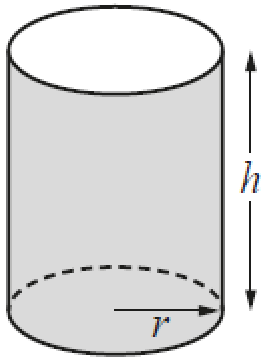
$$y - \ln(8) + \frac{4}{3} = 1$$

$$y = \ln(8) - \frac{1}{3}$$

$\left(0, \ln(8) - \frac{1}{3}\right)$ [1]

(6 marks)

4



A container is a circular cylinder, open at one end, with a base radius of r cm and a height of h cm. The volume of the container is 1000 cm^3. Given that r and h can vary and that the total outer surface area of the container has a minimum value, find this value.

Answer

The volume of a cylinder is $V = \pi r^2 h$.

$$1000 = \pi r^2 h$$

Find h in terms of r.

$$h = \frac{1000}{\pi r^2}$$

[1]

The cylinder is open at one end so its surface area is $S = \pi r^2 + 2\pi rh$, so substitute h into this so the surface area is in terms of r only.

$$S = \pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

[1]

Simplify.

$$S = \pi r^2 + \frac{2000}{r}$$

[1]

Differentiate in order to find the minimum surface area. First, rewrite S as powers of r.

$$S = \pi r^2 + 2000r^{-1}$$

$$\frac{dS}{dr} = 2\pi r - 2000r^{-2}$$

[1]

When the surface area is at its minimum, $\frac{dS}{dr} = 0$.

$$\frac{dS}{dr} = 2\pi r - 2000r^{-2} = 0$$

[1]

Solve to find r.

$$2\pi r - \frac{2000}{r^2} = 0$$

Multiply through by r^2.

$$2\pi r^3 - 2000 = 0$$

$$2\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{1000}{\pi}}$$

[1]

Substitute this value into the formula for S.

$$S = \pi\left(\sqrt[3]{\frac{1000}{\pi}}\right)^2 + \frac{2000}{\left(\sqrt[3]{\frac{1000}{\pi}}\right)}$$

$$S = 439.377\,566\ldots$$

[1]

The minimum surface area is $439\ \text{cm}^2$ (3 s.f.) [1]

(8 marks)

5 (a) Find the x-coordinates of the stationary points of the curve $y = e^{3x}(2x+3)^6$.

Answer

Differentiate using the product rule, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

$u = e^{3x}$ and $v = (2x+3)^6$

$$\frac{du}{dx} = 3e^{3x}$$

[1]

Use the chain rule for $\frac{dv}{dx}$.

$$\frac{dv}{dx} = 6(2x+3)^5 \times 2 = 12(2x+3)^5$$

[1]

Apply product rule.

$$\frac{dy}{dx} = e^{3x} \times 12(2x+3)^5 + (2x+3)^6 \times 3e^{3x}$$

[1]

Factorise a common factor of $3e^{3x}$.

$$\frac{dy}{dx} = 3e^{3x}\left(4(2x+3)^5 + (2x+3)^6\right)$$

Factorise another common factor of $(2x+3)^5$.

$$\frac{dy}{dx} = 3e^{3x}(2x+3)^5(2x+7)$$

[1]

Stationary points occur when $\frac{dy}{dx} = 0$.

$$0 = 3e^{3x}(2x + 3)^5(2x + 7)$$

Solve.

$$3e^{3x} = 0 \quad \text{no solutions}$$

$$(2x+3)^5 = 0 \quad x = -\frac{3}{2}$$

$$(2x+7) = 0 \quad x = -\frac{7}{2}$$

[1]

$x = -1.5$ or $x = -3.5$ [1]

(6 marks)

(b) A curve has equation $y = f(x)$ and has exactly two stationary points. Given that $f''(x) = 4x - 7$, $f'(0.5) = 0$ and $f'(3) = 0$, use the second derivative test to determine the nature of each of the stationary points of this curve.

Answer

Substituting each value of *x* into the second derivative.

$$f''(0.5) = 4(0.5) - 7 = -5, \ -5 < 0$$

$$f''(3) = 4(3) - 7 = 5, \ 5 > 0$$

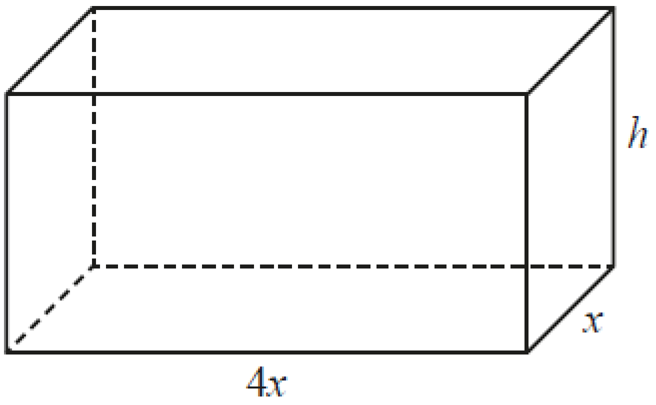
A maximum point occurs when $f''(x) < 0$ and a minimum point occurs when $f''(x) > 0$.

Maximum point at $x = 0.5$ [1]

Minimum point at $x = 3$ [1]

(2 marks)

(c) In this question all lengths are in centimetres.



The diagram shows a solid cuboid with height h and a rectangular base measuring $4x$ by x. The volume of the cuboid is $40\ \text{cm}^3$. Given that x and h can vary and that the surface area of the cuboid has a minimum value, find this value.

Answer

The volume of the cuboid is

$$4x^2h = 40$$

Write h in terms of r.

$$x^2h = 10$$

$$h = \frac{10}{x^2}$$

[1]

The surface area of the cuboid is

$$S = 2(4x^2) + 2xh + 2(4xh)$$

$$S = 8x^2 + 10xh$$

Substitute in the expression of h.

$$S = 8x^2 + 10x\left(\frac{10}{x^2}\right)$$

[1]

Simplify.

$$S = 8x^2 + \frac{100}{x}$$

Differentiate S to find the minimum value but first rewrite as powers of x.

$$S = 8x^2 + 100x^{-1}$$

$$\frac{dS}{dt} = 16x - 100x^{-2}$$

$$\frac{dS}{dx} = 16x - \frac{100}{x^2}$$

[1]

The minimum value occurs at $\frac{dS}{dx} = 0$.

$$0 = 16x - \frac{100}{x^2}$$

$$\frac{100}{x^2} = 16x$$

$$100 = 16x^3$$

$$\frac{25}{4} = x^3$$

$$\sqrt[3]{\frac{25}{4}} = x$$

[1]

Substituting this value into the expression for the surface area.

$$S = 8\left(\sqrt[3]{\frac{25}{4}}\right)^2 + \frac{100}{\sqrt[3]{\frac{25}{4}}}$$

81.4 cm^2 (3 s.f) [1]

(5 marks)

6 (a) Find the equation of the tangent to the curve $2y = \tan 2x + 7$ at the point where $x = \frac{\pi}{8}$.

Give your answer in the form $ax - y = \frac{\pi}{b} + c$, where a, b and c are integers.

Answer

First, divide both sides of the given equation by 2 to make y the subject.

$$y = \frac{1}{2}\tan 2x + \frac{7}{2}$$

Differentiate to find the gradient function.

$$\frac{dy}{dx} = \sec^2 2x$$

[1]

Find the gradient when $x = \frac{\pi}{8}$.

$$\frac{dy}{dx} = \sec^2\left(\frac{2\pi}{8}\right)$$

$$\frac{dy}{dx} = 2$$

[1]

Find y when $x = \frac{\pi}{8}$ by substituting into the original equation.

$$2y = \tan\left(2 \times \frac{\pi}{8}\right) + 7$$

$$y = 4$$

[1]

Use the gradient 2 and point $\left(\frac{\pi}{8}, 4\right)$ to find the equation of the line. Using point–gradient form, "$y - y_1 = m(x - x_1)$",

$$y - 4 = 2\left(x - \frac{\pi}{8}\right)$$

[1]

Expand the bracket.

$$y - 4 = 2x - \frac{\pi}{4}$$

Subtract y and add $\frac{\pi}{4}$ to both sides to obtain the required form $ax - y = \frac{\pi}{b} + c$.

$$\mathbf{2x - y = \frac{\pi}{4} - 4}$$ [1]

(5 marks)

(b) This tangent intersects the x-axis at P and the y-axis at Q. Find the length of PQ.

Answer

From part (a), the equation of the tangent is $2x - y = \frac{\pi}{4} - 4$.

The tangent will intersect the x-axis when $y = 0$, so substitute $y = 0$ to find the x coordinate P.

$$2x - 0 = \frac{\pi}{4} - 4$$

$$x = \frac{\pi}{8} - 2$$

$$P\left(\frac{\pi}{8} - 2,\ 0\right)$$

The tangent will intersect the y-axis when $x = 0$, so substitute $x = 0$ to find the y coordinate of Q.

$$0 - y = \frac{\pi}{4} - 4$$

$$y = 4 - \frac{\pi}{4}$$

$$Q\left(0,\ 4-\frac{\pi}{4}\right)$$

Calculate the distance between the points to find the length of PQ using Pythagoras' theorem.

$$PQ=\sqrt{\left(\frac{\pi}{8}-2\right)^2+\left(\frac{\pi}{4}-4\right)^2}$$

[1]

$$PQ=3.594...$$

The length of PQ is 3.59 (3 s.f.) [1]

(2 marks)

7 (a) $y = x\sqrt{x+2}$ Given that , show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Ax+B}{2\sqrt{x+2}}$, where A and B are constants.

Answer

Differentiate using product rule, $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$, and chain rule.

$$u = x \quad v = (x+2)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}(x+2)^{-\frac{1}{2}}$$

[1]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1)(x+2)^{\frac{1}{2}} + (x)\left(\frac{1}{2}(x+2)^{-\frac{1}{2}}\right)$$

[1]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x(x+2)^{-\frac{1}{2}} + (x+2)^{\frac{1}{2}}$$

[1]

Simplify by taking factors of $\frac{1}{2}$ and $(x+2)^{-\frac{1}{2}}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(x+2)^{-\frac{1}{2}}[x + 2(x+2)]$$

[1]

Rewrite as a fraction.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{x+2}}\right)(3x+4)$$

Simplify to get in the required form.

$$\frac{dy}{dx} = \frac{3x+4}{2\sqrt{x+2}}$$ [1]

(5 marks)

(b) Find the exact coordinates of the stationary point of the curve $y = x\sqrt{x+2}$.

Answer

From part (a), $\frac{dy}{dx} = \frac{3x+4}{2\sqrt{x+2}}$.

Stationary points occur when the gradient is 0, so substitute $\frac{dy}{dx} = 0$.

$$\frac{3x+4}{2\sqrt{x+2}} = 0$$

Rearrange and solve.

$$3x+4=0$$

[1]

$$x = -\frac{4}{3}$$

[1]

Substitute into the equation of the curve to find the y coordinate.

$$y = \left(-\frac{4}{3}\right)\left(\sqrt{-\frac{4}{3}+2}\right)$$

$$y = -\frac{4\sqrt{2}}{3\sqrt{3}}$$

Rationalise the denominator.

$$y = -\frac{4\sqrt{2}}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$y = -\frac{4\sqrt{6}}{9}$$

$\left(-\frac{4}{3}, -\frac{4\sqrt{6}}{9}\right)$ [1]

(3 marks)

(c) Determine the nature of this stationary point.

Answer

The stationary point occurs at $x = -\frac{4}{3}$.

To find the nature of this point, we can find the gradient, *just* either side of $x = -\frac{4}{3}$.

When $x = -1$, (to the right of the stationary point),

$$\frac{dy}{dx} = \frac{3(-1)+4}{2\sqrt{(-1)+2}}$$

$$\frac{dy}{dx} = \frac{1}{2}$$

When $x = -\frac{5}{3}$, (to the left of the stationary point),

$$\frac{dy}{dx} = \frac{3\left(-\frac{5}{3}\right)+4}{2\sqrt{\left(-\frac{5}{3}\right)+2}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{3}}{2}$$

[1]

Since the gradient changes from negative to positive through the point where $x = -\frac{4}{3}$

so the curve is "$\cup$-shaped".

$$\therefore \left(-\frac{4}{3}, -\frac{4\sqrt{6}}{9}\right)$$ **is a minimum point** [1]

(2 marks)

8 (a) Differentiate $y = \tan(x+4) - 3\sin x$ with respect to x.

Answer

Differentiate the two terms separately.

Differentiate $\tan(x+4)$ with respect to x.

$$\sec^2(x+4)$$

Differentiate $-3\sin x$ with respect to x.

$$-3\cos x$$

either correct [1]

Put both parts together.

$\sec^2(x+4) - 3\cos x$ [1]

(2 marks)

(b) Variables x and y are such that $y = \dfrac{\ln(2x+5)}{2e^{3x}}$. Use differentiation to find the approximate change in y as x increases from 1 to $1 + h$, where h is small.

Answer

Differentiate $\dfrac{\ln(2x+5)}{2e^{3x}}$ using the quotient rule.

$$u = \ln(2x+5) \text{ and } v = 2e^{3x}$$

$$\frac{du}{dx} = \frac{2}{2x+5} \text{ and } \frac{dv}{dx} = 6e^{3x}$$

for $\dfrac{2}{2x+5}$ [1]

for $6e^{3x}$ [1]

$$\frac{dy}{dx} = \frac{\left(2e^{3x} \times \frac{2}{2x+5}\right) - (6e^{3x} \times \ln(2x+5))}{(2e^{3x})^2}$$

[1]

$$\frac{dy}{dx} = \frac{\left(2e^{3x} \times \frac{2}{2x+5}\right) - (6e^{3x} \times \ln(2x+5))}{4e^{6x}}$$

[1]

Substitute $x = 1$ into the equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\left(2e^{3} \times \frac{2}{7}\right) - (6e^{3} \times \ln 7)}{4e^{6}}$$
$$= -0.1382093041$$

As h is small, use "$\text{change in } y = \text{gradient} \times \text{change in } x$".

change in $y = -0.1382093041 \times h$

[1]

The approximate change in y is $-0.14h$ [1]

(6 marks)

9 (a) It is given that $y = \dfrac{\tan 3x}{\sin x}$.

Find the exact value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{3}$.

Answer

This function is in the form $y = \dfrac{u}{v}$ so use the quotient rule, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

$$u = \tan 3x \qquad v = \sin x$$

$$\frac{du}{dx} = 3\sec^2 3x \qquad \frac{dv}{dx} = \cos x$$

for $3\sec^2 3x$ [1]

Substitute these into the quotient rule.

$$\frac{dy}{dx} = \frac{\sin x \times 3\sec^2 3x - \tan 3x \times \cos x}{\sin^2 x}$$

using the quotient rule [1]
all terms correct [1]

Use $\sec x = \dfrac{1}{\cos x}$ to evaluate $\sec^2 3x$.

$$\frac{dy}{dx} = \frac{\sin x \times 3\left(\dfrac{1}{\cos^2 3x}\right) - \tan 3x \times \cos x}{\sin^2 x}$$

Substitute $x = \dfrac{\pi}{3}$.

$$\frac{dy}{dx} = 2\sqrt{3} \text{ [1]}$$

(4 marks)

(b) Hence find the approximate change in y as x increases from $\frac{\pi}{3}$ to $\frac{\pi}{3}+h$, where h is small.

Answer

As h is small, use "$\text{change in } y = \text{gradient} \times \text{change in } x$".

From part (a), at $x = \frac{\pi}{3}, \frac{dy}{dx} = 2\sqrt{3}$.

$$\text{change in } y = 2\sqrt{3} \times h = 3.464\,101...$$

The approximate change in y is $3.5h$ [1]

(1 mark)

(c) Given that x is increasing at the rate of 3 units per second, find the corresponding rate of change in y when $x = \frac{\pi}{3}$, giving your answer in its simplest surd form.

Answer

This is a connected rates of change question, so use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

The question states that x is increasing at the rate of 3 units per second.

$$\frac{dx}{dt} = 3$$

From part (a), at $x = \frac{\pi}{3}$.

$$\frac{dy}{dx} = 2\sqrt{3}$$

[1]

$$\therefore \frac{dy}{dt} = 2\sqrt{3} \times 3$$

$$\frac{dy}{dt} = 6\sqrt{3}$$ [1]

(2 marks)

10 (a) It is given that $y = \ln(\sin x + 3\cos x)$ for $0 < x < \frac{\pi}{2}$.

Find $\frac{dy}{dx}$.

Answer

Use chain rule to differentiate $\ln(f(x))$.

$$\frac{dy}{dx} = \frac{1}{(\sin x + 3\cos x)} \times (\cos x - 3\sin x)$$

differentiating ln [1]
chain rule [1]

Simplify.

$$\frac{dy}{dx} = \frac{\cos x - 3\sin x}{\sin x + 3\cos x}$$ [1]

(3 marks)

(b) Find the value of x for which $\frac{dy}{dx} = -\frac{1}{2}$.

Answer

Set $\frac{dy}{dx}$ from part (a) equal to $-\frac{1}{2}$.

$$-\frac{1}{2} = \frac{\cos x - 3\sin x}{\sin x + 3\cos x}$$

Multiply both sides by the denominators.

$$-(\sin x + 3\cos x) = 2\cos x - 6\sin x$$

Expand the brackets.

$$-\sin x - 3\cos x = 2\cos x - 6\sin x$$

[1]

Collect like terms.

$$5\sin x = 5\cos x$$

Dividing both sides by $5\cos x$ and using the identity $\frac{\sin x}{\cos x} = \tan x$.

$$\tan x = 1$$

[1]

Solving for x using the range given in part (a).

$x = \frac{\pi}{4}$ [1]

(3 marks)

11 (a) Given that $y = \dfrac{e^{2x-3}}{x^2+1}$, find $\dfrac{dy}{dx}$.

Answer

Differentiate the equation using the quotient rule and chain rule.

$$u = e^{2x-3} \qquad v = x^2 + 1$$

$$\frac{du}{dx} = 2e^{2x-3} \quad \frac{dv}{dx} = 2x$$

Substitute into the quotient rule, $\dfrac{dy}{dx} = \dfrac{(v)\left(\dfrac{du}{dx}\right) - (u)\left(\dfrac{dv}{dx}\right)}{v^2}$.

$$\frac{dy}{dx} = \frac{(x^2+1)(2e^{2x-3}) - (2x)(e^{2x-3})}{(x^2+1)^2}$$

$2e^{2x-1}$ [1]

[1]

Split into two fractions and simplify.

$$\frac{dy}{dx} = \frac{(x^2+1)(2e^{2x-3})}{(x^2+1)^2} - \frac{(2x)(e^{2x-3})}{(x^2+1)^2}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{(2e^{2x-3})}{(x^2+1)} - \frac{(2xe^{2x-3})}{(x^2+1)^2}}$$ [1]

(3 marks)

(b) Hence, given that y is increasing at the rate of 2 units per second, find the exact rate of change of x when $x = 2$.

Answer

This is a connected rates of change question, so we shall need chain rule.

$$\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt}$$

y is increasing at a rate of 2 units per second, i.e. $\frac{dy}{dt} = 2$. Substitute $x = 2$ into the equation for $\frac{dy}{dx}$ from part (a).

$$\frac{dy}{dx} = \frac{2e^{2\times 2-3}}{2^2+1} - \frac{2\times 2\times e^{2\times 2-3}}{(2^2+1)^2}$$
$$\frac{dy}{dx} = \frac{2e}{5} - \frac{4e}{25}$$
$$\frac{dy}{dx} = \frac{6e}{25}$$

[1]

Keep this in terms of e as the question asks for an exact answer. Substitute into chain rule and solve for $\frac{dt}{dx}$.

$$\frac{6e}{25} = \frac{dt}{dx} \times 2$$
$$\frac{dt}{dx} = \frac{6e}{50}$$

We want to find $\frac{dx}{dt}$ so we need the reciprocal of $\frac{dt}{dx}$.

$$\frac{dx}{dt} = \frac{50}{6e}$$

$\frac{dx}{dt} = \frac{25}{3e}$ [1]

(3 marks)

12 A sphere of radius r cm and volume V cm^3 is increasing in size with time t seconds. The volume increases at a constant rate of 24 cm^3 s^{-1}.

Find the exact rate at which the radius is increasing when the sphere reaches a volume of $\frac{32\pi}{3}$ cm^3.

Answer

The volume of a sphere is given in the exam

$$V = \frac{4}{3}\pi r^3$$

This question requires connect rates of change (an application of the chain rule)

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Find $\frac{dV}{dr}$ by differentiating $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2)$$
$$= 4\pi r^2$$

[B1]

You know $\frac{dV}{dt} = 24$ from the question

Substitute $\frac{dV}{dt} = 24$ and $\frac{dV}{dr} = 4\pi r^2$ into $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$24 = 4\pi r^2 \times \frac{dr}{dt}$$

[M1]

Mark Scheme and Guidance

This mark is for using a correct connected rate formula, e.g. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$, with at least one correct substitution into the formula.

The question wants you to find $\frac{\mathrm{d}r}{\mathrm{d}t}$ so make $\frac{\mathrm{d}r}{\mathrm{d}t}$ the subject

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{24}{4\pi r^2} = \frac{6}{\pi r^2}$$

You need the value of $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $V = \frac{32\pi}{3}$ but $\frac{\mathrm{d}r}{\mathrm{d}t}$ is in terms of r not V

Use $V = \frac{4}{3}\pi r^3$ to find r when $V = \frac{32\pi}{3}$

$$\frac{32\pi}{3} = \frac{4}{3}\pi r^3$$
$$32\pi = 4\pi r^3$$
$$8 = r^3$$
$$r = 2$$

[A1]

Substitute $r = 2$ into $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{6}{\pi(2)^2} = \frac{6}{4\pi} = \frac{3}{2\pi}$$

The units are cm for r and seconds for t, so cm/s for $\frac{\mathrm{d}r}{\mathrm{d}t}$

$\frac{3}{2\pi}$ **cm s^{-1}**

[A1]

Mark Scheme and Guidance

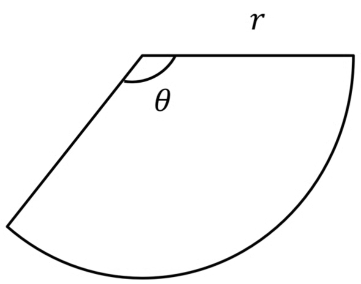
The correct exact value gets this mark, even without the units.

Examiner Tips and Tricks

Be careful: not all questions forgive units (for example, questions that ask for answers in different units or ask for an interpretation).

(4 marks)

13 (a) A sector from a circle of radius r has an internal angle of θ radians, as shown below.



The perimeter of the sector is 4 units and the area of the sector is A square units.

Show that $A = \dfrac{8\theta}{(2+\theta)^2}$.

Answer

The formula for the length of the arc in radians is $r\theta$

Find an expression for the total perimeter and set it equal to 4

$$r + r\theta + r = 4$$

$$2r + r\theta = 4 \qquad ①$$

[B1]

Mark Scheme and Guidance

This mark is for any correct perimeter equation that uses 4 and $r\theta$.

The formula for the area of the arc in radians is $\frac{1}{2}r^2\theta$

Find an expression for the area A

$$A = \frac{1}{2}r^2\theta \qquad ②$$

Eliminate r, e.g. make r the subject of equation 1 by first factorising it out

$$r(2 + \theta) = 4$$

$$r = \frac{4}{2 + \theta}$$

[M1]

Then substitute this into equation 2

$$A = \frac{1}{2}\left(\frac{4}{2 + \theta}\right)^2 \times \theta$$

Simplify

$$A = \frac{1}{2} \times \frac{16}{(2 + \theta)^2} \times \theta$$

$$A = \frac{8\theta}{(2 + \theta)^2}$$

[A1]

Mark Scheme and Guidance

The last mark is for showing the steps that get you from the substitution of r into A to the answer given (not for writing out the "show that" answer).

(3 marks)

(b) Show that $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = \dfrac{p(q-\theta)}{(2+\theta)^3}$ where p and q are constants to be found.

Answer

To differentiate $A = \dfrac{8\theta}{(2+\theta)^2}$ you need the quotient rule

Substitute $u = 8\theta$ and $v = (2+\theta)^2$ into $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

You can use the chain rule to differentiate $v = (2+\theta)^2$ (or expand the brackets then differentiate)

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}x} &= 2(2+\theta)^1 \times 1 \\ &= 2(2+\theta)\end{aligned}$$

The quotient rule gives

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{(2+\theta)^2(8) - 8\theta \times 2(2+\theta)}{[(2+\theta)^2]^2}$$

[M1 A1]

Factorise out a $(2+\theta)$ from top and bottom of the fraction, then cancel

$$\frac{dA}{d\theta} = \frac{8(2+\theta)^2 - 16\theta(2+\theta)}{(2+\theta)^4}$$

$$= \frac{(2+\theta)[8(2+\theta) - 16\theta]}{(2+\theta)^4}$$

$$= \frac{\cancel{(2+\theta)}[8(2+\theta) - 16\theta]}{(2+\theta)^{\cancel{4}3}}$$

$$= \frac{8(2+\theta) - 16\theta}{(2+\theta)^3}$$

Write this in the form $\frac{dA}{d\theta} = \frac{p(q-\theta)}{(\theta+2)^3}$ by simplifying and factorising the numerator

$$\frac{dA}{d\theta} = \frac{16 + 8\theta - 16\theta}{(2+\theta)^3}$$

$$= \frac{16 - 8\theta}{(2+\theta)^3}$$

$$= \frac{8(2-\theta)}{(2+\theta)^3}$$

$$\frac{dA}{d\theta} = \frac{8(2-\theta)}{(2+\theta)^3}$$

[A1]

(3 marks)

(c) Find the maximum area of the sector. You must show that this area is a maximum.

Answer

For the area to be at its maximum point, $\frac{dA}{d\theta} = 0$

$$\frac{8(2-\theta)}{(2+\theta)^3} = 0$$

This means the numerator must be equal to zero (then solve for θ)

$$8(2-\theta) = 0$$

$$\theta = 2$$

[B1]

This is the angle, not the area, so substitute $\theta = 2$ into A

$$A = \frac{8\times 2}{(2+2)^2}$$
$$= \frac{16}{4^2}$$
$$= 1$$

[B1]

This area, $A = 1$, could be a maximum area or a minimum area

Method 1

You can use a second-derivative test to show it is a maximum area

Substitute $u = 8(2-\theta) = 16 - 8\theta$ and $v = (2+\theta)^3$ into the quotient rule (using the chain rule to find $\frac{\mathrm{d}v}{\mathrm{d}x}$)

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = \frac{(2+\theta)^3(-8) - 8(2-\theta)\times 3(2+\theta)^2\times 1}{[(2+\theta)^3]^2}$$

No major simplifying is needed as you can now substitute $\theta = 2$ into the messy result and check the sign

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = \frac{(2+2)^3(-8) - 8\times 0}{[(2+2)^3]^2}$$
$$= \frac{4^3(-8)}{(4^3)^2}$$
$$= -\frac{8}{4^3} < 0$$

The second derivative is negative at $\theta = 2$ so $A = 1$ is a maximum area

The area of 1 square unit is a maximum area as $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} < 0$ **at** $\theta = 2$

[M1 A1]

Mark Scheme and Guidance

M1: For attempting to find the second derivative using the quotient rule.

A1: For correctly showing that the second derivative is negative at $\theta = 2$ (no formal conclusion sentence needed).

Method 2

You can use a gradient test either side of $\theta = 2$ to show it is a maximum area

Substitute, for example, $\theta = 1.9$ into $\frac{dA}{d\theta}$ and just look at its sign (no big simplifications are needed)

$$\frac{dA}{d\theta} = \frac{8(2-1.9)}{(2+1.9)^3} = \frac{8\times 0.1}{(3.9)^3} > 0$$

Substitute, for example, $\theta = 2.1$ into $\frac{dA}{d\theta}$ and again just look at its sign

$$\frac{dA}{d\theta} = \frac{8(2-2.1)}{(2+2.1)^3} = \frac{8\times(-0.1)}{(4.1)^3} = -\frac{8\times 0.1}{(4.1)^3} < 0$$

The area of 1 square unit is a maximum area as the gradient $\frac{dA}{d\theta}$ goes from positive to negative

[M1 A1]

Mark Scheme and Guidance

M1: For an attempt to find the value of $\frac{dA}{d\theta}$ either side of $\theta = 2$.

A1: For correctly showing that the gradient goes from positive to negative from left of $\theta = 2$ to right of $\theta = 2$ (no formal conclusion sentence needed).

(4 marks)

14 A curve is given by $y = \ln(1 + x^4)$.

Use calculus to find the approximate change in y as x increases from -1 to $-1 + k$ where k is small.

Answer

In general, the approximate change in y as x increases from a to $a + h$ where h is small is $\frac{dy}{dx} \times h$ where $\frac{dy}{dx}$ is at $x = a$

Substitute $a = -1$ and $h = k$ into the expression above

$$\frac{dy}{dx} \times k \text{ where } \frac{dy}{dx} \text{ is at } x = -1$$

Find $\frac{dy}{dx}$ using the chain rule, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $u = 1 + x^4$

$$y = \ln u \qquad u = 1 + x^4$$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 4x^3$$

This gives

$$\frac{dy}{dx} = \frac{1}{u} \times 4x^3$$

$$= \frac{1}{1 + x^4} \times 4x^3$$

$$= \frac{4x^3}{1 + x^4}$$

[M1 A1]

Substitute $x = -1$ into $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{4(-1)^3}{1+(-1)^4} = -2$$

[M1]

Substitute $\frac{dy}{dx} = -2$ into $\frac{dy}{dx} \times k$

$-2k$

[A1]

Mark Scheme and Guidance

M1: For having $\frac{1}{1+x^4}$ in your expression for $\frac{dy}{dx}$.

A1: For the fully correct expression for $\frac{dy}{dx}$.

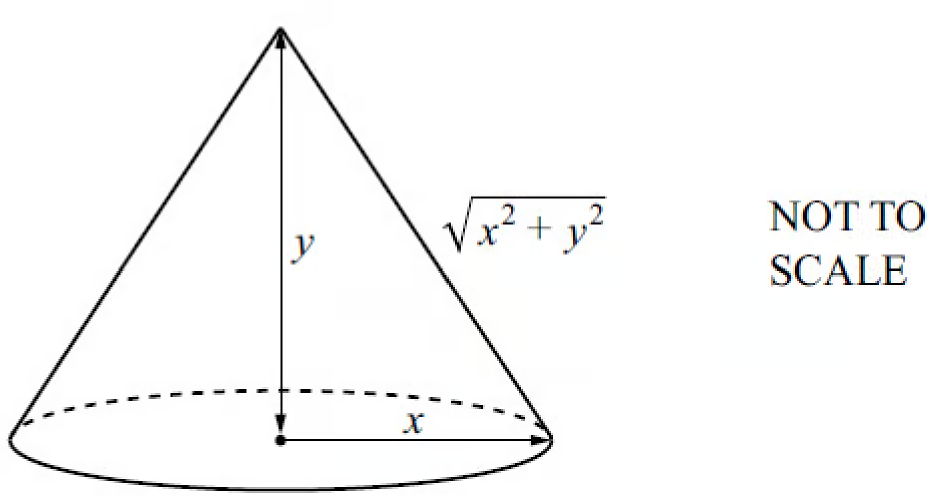
M1: For substituting $x = -1$ into your expression for $\frac{dy}{dx}$.

A1: For $-2k$ as the final answer.

(4 marks)

Very Hard Questions

1 (a) In this question, all lengths are in centimetres.



The diagram shows a cone of base radius x, height y and sloping edge $\sqrt{x^2+y^2}$. The volume of the cone is 10π cm^3.

Show that the curved surface area, S, of the cone is given by $S = \dfrac{\pi\sqrt{x^6+900}}{x}$.

Answer

We know that the curved surface area of a cone, S, is $\pi r l$.
r in this case is actually x and the slant height l is given as $\sqrt{x^2+y^2}$. But the expression given for the curved surface are is in terms of x only.
Therefore we need to find an expression for y in terms of x.

We are given a value for the volume, '10π'. Therefore equate 10π to the formula for volume of a cone in terms of the radius x and height y.

$$10\pi = \frac{1}{3}\pi x^2 y$$

Rearrange to get an expression for y in terms of x.

$$y = \frac{10\pi}{\frac{1}{3}\pi x^2}$$

$$= \frac{30}{x^2}$$

[1]

Now substitute $x = r$ and $l = \sqrt{x^2 + y^2}$ where $y = \dfrac{30}{x^2}$ into $S = \pi r l$.

$$S = \pi x \sqrt{x^2 + \left(\frac{30}{x^2}\right)^2}$$

[1]

This is starting to approach something that looks like the final answer. To get to the final answer, start by expanding the bracket inside the square root.

$$S = \pi x \sqrt{x^2 + \frac{900}{x^4}}$$

Note that there's no fraction inside the square root of the final answer. Rewrite '$x^2 + \dfrac{900}{x^4}$' as a single fraction.

$$S = \pi x \sqrt{\frac{x^6 + 900}{x^4}}$$

Now take the square root of the denominator of the fraction.

$$S = \pi x \frac{\sqrt{x^6 + 900}}{x^2}$$

Cancel the x and the x^2.

$$S = \pi \cancel{x} \frac{\sqrt{x^6 + 900}}{\cancel{x}^2}$$

$$S = \frac{\pi\sqrt{x^6 + 900}}{x}$$ [1]

(3 marks)

(b) Given that x can vary and that S has a minimum value, find the value of x for which S is a minimum.

Answer

We will need to differentiate S and then equate the derivative to 0 to find the minimum value of x.
As differentiation will be involved, rewrite the square root as a fractional power.

$$S = \frac{\pi\sqrt{x^6+900}}{x} = \frac{\pi(x^6+900)^{\frac{1}{2}}}{x}$$

To differentiate S we need to use the quotient rule.

$$S = \frac{\pi(x^6+900)^{\frac{1}{2}}}{x} = \frac{u}{v}, \text{ where } u = \pi(x^6+900)^{\frac{1}{2}} \text{ and } v = x$$

From here we need to find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. However we need to use the chain rule to find $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$u = \pi(x^6+900)^{\frac{1}{2}}, \text{ let } u = \pi\, w^{\frac{1}{2}} \text{ and } w = x^6+900$$

Now differentiate *u* and *w*.

$$\frac{\mathrm{d}u}{\mathrm{d}w} = \pi \times \frac{1}{2} w^{-\frac{1}{2}}, \quad \frac{\mathrm{d}w}{\mathrm{d}x} = 6x^5$$

And apply the chain rule, in this case $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}x}$.

$$\frac{du}{dx} = \pi\frac{1}{2}w^{-\frac{1}{2}} \times 6x^5$$

$$= \pi\frac{1}{2}(x^6+900)^{-\frac{1}{2}} \times 6x^5$$

$\pi\frac{1}{2}(x^6+900)^{-\frac{1}{2}}$ [1]

$$= 3\pi x^5(x^6+900)^{-\frac{1}{2}}$$

[1]

Now we are ready to go back to $S = \frac{u}{v}$ where $u = \pi\sqrt{x^6+900}$ and $v = x$.

$$u = \pi(x^6+900)^{\frac{1}{2}} \qquad \text{and} \qquad v = x$$

$$\frac{du}{dx} = 3\pi x^5(x^6+900)^{-\frac{1}{2}} \qquad \frac{dv}{dx} = 1$$

Apply the quotient rule, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

$$\frac{dS}{dx} = \frac{x\left(3\pi x^5(x^6+900)^{-\frac{1}{2}}\right) - \pi(x^6+900)^{\frac{1}{2}}}{x^2}$$

[1]

At the minimum value of S, $\frac{dS}{dx} = 0$, so equate this derivative to 0.

$$\frac{x\left(3\pi x^5(x^6+900)^{-\frac{1}{2}}\right) - \pi(x^6+900)^{\frac{1}{2}}}{x^2} = 0$$

Solve – start by multiplying by the denominator, x^2.

$$x\left(3\pi x^5(x^6+900)^{-\frac{1}{2}}\right)-\pi(x^6+900)^{\frac{1}{2}}=0$$

[1]

Simplify.

$$3\pi x^6(x^6+900)^{-\frac{1}{2}}-\pi(x^6+900)^{\frac{1}{2}}=0$$

Remove the negative power.

$$\frac{3\pi x^6}{(x^6+900)^{\frac{1}{2}}}-\pi(x^6+900)^{\frac{1}{2}}=0$$

$$\text{or} \quad \frac{3\pi x^6}{\sqrt{x^6+900}}-\pi\sqrt{x^6+900}=0$$

Multiply by $\sqrt{x^6+900}$.

$$3\pi x^6-\pi(x^6+900)=0$$

Divide by π and complete the solution.

$$3x^6-(x^6+900)=0$$
$$3x^6-x^6-900=0$$
$$2x^6=900$$
$$x^6=450$$

$x=\sqrt[6]{450}=2.768229457\ldots$ **or** 2.77 (3 s.f.) [1]

Note that x is a length therefore $x \neq -\sqrt[6]{450}$

(5 marks)

2 Variables x and y are such that $y=\dfrac{e^{3x}\sin x}{x^2}$.Use differentiation to find the approximate change in y as x increases from 0.5 to $0.5+h$, where h is small.

Answer

Recall that

$$\frac{d}{dx}(e^{3x}) = 3e^{3x}$$

[1]

Use the product rule to differentiate the numerator and then apply the quotient rule.

When you differentiate the numerator using the product rule you will get

$$e^{3x}\cos x + 3e^{3x}\sin x$$

[1]

$$\frac{d}{dx}\left(\frac{e^{3x}\sin x}{x^2}\right) = \frac{x^2(e^{3x}\cos x + 3e^{3x}\sin x) - 2x(e^{3x}\sin x)}{(x^2)^2}$$

[1]

$$\frac{d}{dx}\left(\frac{e^{3x}\sin x}{x^2}\right) = \frac{x^2e^{3x}\cos x + 3x^2e^{3x}\sin x - 2xe^{3x}\sin x}{x^4}$$

[1]

Evaluate the derivative above when $x = 0.5$ using the radian mode on your calculator.

$$\frac{d}{dx}\left(\frac{e^{3x}\sin x}{x^2}\right) = 7.13766\ldots$$

[1]

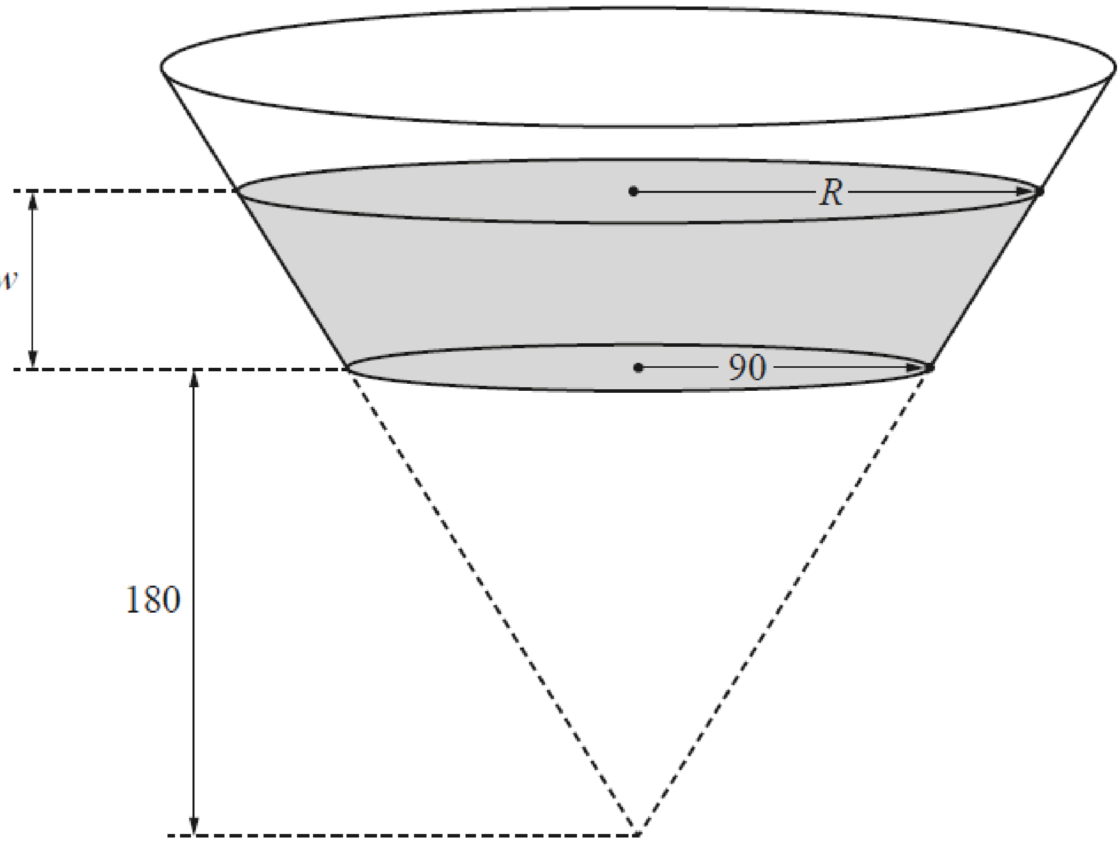
As h is small, use "Change in y = Gradient × Change in x".

$$7.14 \times h$$

Approximate change in y is $7.14h$ [1]

(6 marks)

3 (a) In this question all lengths are in centimetres. The volume, V, of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$



The diagram shows a large hollow cone from which a smaller cone of height 180 and base radius 90 has been removed. The remainder has been fitted with a circular base of radius 90 to form a container for water. The depth of water in the container is w and the surface of the water is a circle of radius R.

Find an expression for R in terms of w and show that the volume V of the water in the container is given by $V = \frac{\pi}{12}(w + 180)^3 - 486000\pi$.

Answer

We actually need to work with a 'third' cone – formed from the larger hollow cone described in the question but with a height that is determined by the (higher) water level. All three cones are similar and since the radius of the smaller cone is half its own height, the radius of the 'third' cone will be half its own height too.

$$\therefore R = \frac{1}{2}(w + 180)$$

[1]

The volume of the water can be found by subtracting the volume of the smaller cone from that of the 'third' cone.

$$V = \frac{1}{3}\pi\left(\frac{1}{2}(w + 180)\right)^2 (w + 180) - \frac{1}{3}\pi(90)^2(180)$$

[1]

Simplify.

$$V = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2 (w + 180)^2(w + 180) - \frac{1}{3}\pi(90)^2(180)$$

$$V = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\pi(w + 180)^3 - 486000\pi$$

$$V = \frac{\pi}{12}(w + 180)^3 - 486000\pi$$ [1]

(3 marks)

(b) Water is poured into the container at a rate of $10\,000 \text{ cm}^3\text{s}^{-1}$. Find the rate at which the depth of the water is increasing when $w = 10$.

Answer

This is a connected rates of change question, we will need to use chain rule. At time t,

$$\frac{dw}{dt} = \frac{dw}{dV} \times \frac{dV}{dt}$$

The rate at which the volume (V) is changing with respect to the height of the water (w) is

$$\frac{dV}{dw} = 3\frac{\pi}{12}(w + 180)^2$$

[1]

We require $\frac{dw}{dV}$.

$$\frac{dw}{dV} = \frac{1}{\frac{\pi}{4}(w+180)^2}$$

We know the rate at which water is poured into the container with respect to time.

$$\frac{dV}{dt} = 10000$$

Apply chain rule.

$$\frac{dw}{dt} = \frac{1}{\frac{\pi}{4}(w+180)^2} \times 10000$$

[1]

$$\frac{dw}{dt} = \frac{40000}{\pi(w+180)^2}$$

When $w = 10$,

$$\frac{dw}{dt} = \frac{40000}{\pi(10+180)^2}$$

[1]

Simplify and evaluate.

$$\frac{dw}{dt} = \frac{40000}{\pi(190)^2}$$

$$\frac{dw}{dt} = 0.3526...$$

$\frac{dw}{dt} = 0.353 \text{ cm s}^{-1}$ (3 s.f.) [1]

(4 marks)

4 (a) A curve has equation $y = \dfrac{\ln(3x^2 - 5)}{2x + 1}$ for $3x^2 > 5$

Find the equation of the normal to the curve at the point where $x = \sqrt{2}$.

Answer

Differentiate the equation using quotient rule and chain rule.

$$u = \ln(3x^2 - 5) \qquad v = 2x + 1$$

$$\frac{du}{dx} = \left(\frac{1}{3x^2 - 5}\right) \times 6x \qquad \frac{dv}{dx} = 2$$

correct $\dfrac{du}{dx}$ [1]

Use quotient rule, $\dfrac{dy}{dx} = \dfrac{(v)\left(\dfrac{du}{dx}\right) - (u)\left(\dfrac{dv}{dx}\right)}{v^2}$.

$$\frac{dy}{dx} = \frac{\left((2x+1)\left(\dfrac{6x}{3x^2 - 5}\right)\right) - (2)(\ln(3x^2 - 5))}{(2x+1)^2}$$

quotient rule [1]

all terms correct except $\dfrac{6x}{3x^2 - 5}$ [1]

When $x = \sqrt{2}$,

$$y = \frac{\ln(1)}{2\sqrt{2} + 1}$$

$\ln(1) = 0$, therefore when $x = \sqrt{2}$,

$$y = 0$$

[1]

Substitute $x = \sqrt{2}$ into $\frac{dy}{dx}$ to find the gradient of the curve at this point.

$$\frac{dy}{dx} = \frac{(2\sqrt{2}+1)\left(\frac{6\sqrt{2}}{3(\sqrt{2})^2-5}\right) - 2\ln(3(\sqrt{2})^2-5)}{(2\sqrt{2}+1)^2}$$

$$= \frac{\left(\frac{(2\sqrt{2}+1)(6\sqrt{2})}{1}\right) - 2\ln(1)}{(2\sqrt{2}+1)^2}$$

$$= \frac{(2\sqrt{2}+1)(6\sqrt{2})}{(2\sqrt{2}+1)^2}$$

$$= \frac{6\sqrt{2}}{2\sqrt{2}+1}$$

Take the negative reciprocal to find the gradient of the normal.

$$\text{gradient (normal)} = -\frac{2\sqrt{2}+1}{6\sqrt{2}}$$

Substitute this gradient and the point $(\sqrt{2}, 0)$ into the equation of line, $y - y_1 = m(x - x_1)$.

$$y - 0 = -\frac{2\sqrt{2}+1}{6\sqrt{2}}(x - \sqrt{2})$$

[1]

$$y = -\frac{(2\sqrt{2}+1)(x-\sqrt{2})}{6\sqrt{2}}$$

$$y = -0.451\,184\ldots\,x + 0.638\,071\ldots$$

$y = -0.451x + 0.638$ (3 s.f.) [1]

(6 marks)

(b) Find the approximate change in y as x increases from $\sqrt{2}$ to $\sqrt{2}+h$, where h is small.

Answer

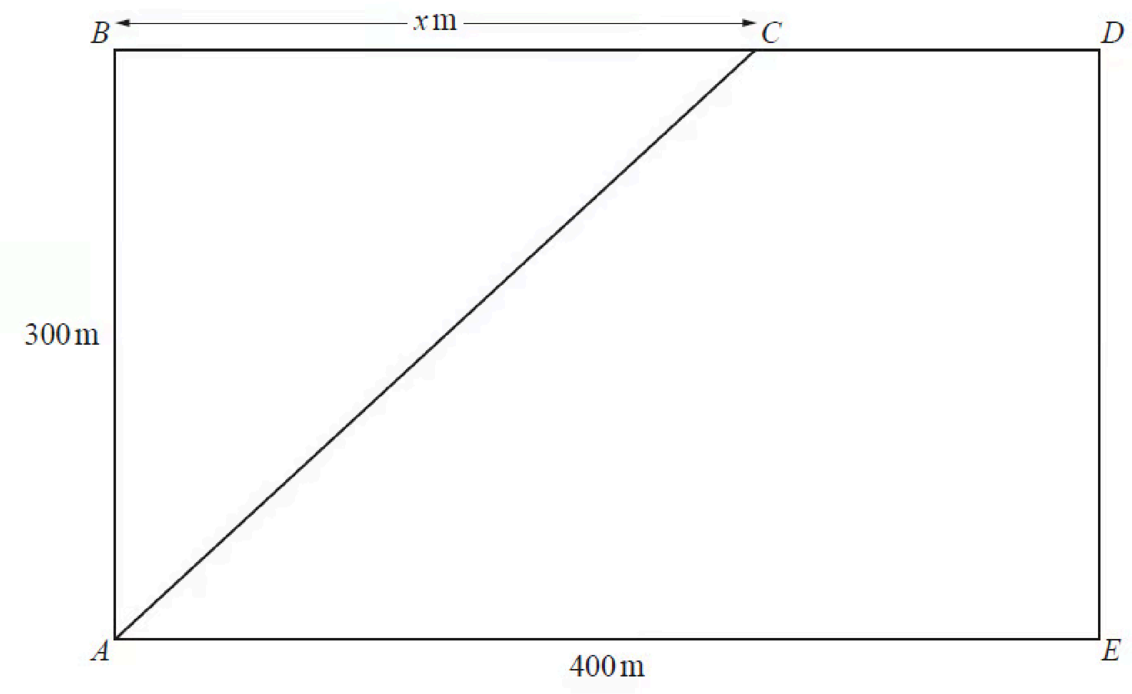
As is small, use "change in y = gradient × change in x".

$$\text{change in } y = \frac{6\sqrt{2}}{2\sqrt{2}+1} \times h$$
$$= 2.216\,388\ldots$$

The approximate change in y is $2.2h$ [1]

(1 mark)

5 (a)



The rectangle $ABCDE$ represents a ploughed field where $AB = 300$ m and $AE = 400$ m. Joseph needs to walk from A to D in the least possible time. He can walk at $0.9\ \text{ms}^{-1}$ on the ploughed field and at $1.5\ \text{ms}^{-1}$ on any part of the path BCD along the edge of the field. He walks from A to C and then from C to D. The distance $BC = x$ m.

Find, in terms of x, the total time, T s, Joseph takes for the journey.

Answer

Use Pythagoras' Theorem to find the length *AC* in terms of x.

$$AC^2 = 300^2 + x^2$$
$$AC = \sqrt{90\,000 + x^2}$$

[1]

Use "$\text{av. speed} = \frac{\text{distance}}{\text{time}}$" (rearranged) to find the time to walk *AC* (speed on the ploughed field).

$$\text{Time for } AC = \frac{\sqrt{90\,000 + x^2}}{0.9} = \frac{10}{9}\sqrt{90\,000 + x^2}$$

[1]

Similarly for *CD* (speed on the path),

$$\text{Time for } CD = \frac{400 - x}{1.5} = \frac{2(400 - x)}{3}$$

Add the times together to find the total time, *T*.

$$T = \frac{10}{9}\sqrt{90\,000 + x^2} + \frac{2(400 - x)}{3}$$

$$T = \frac{6(400 - x) + 10\sqrt{90\ 000 + x^2}}{9}$$ [1]

Equivalent expressions are acceptable

(3 marks)

(b) Given that x can vary, find the value of x for which T is a minimum and hence find the minimum value of T.

Answer

Rewrite *T* to make it easier to differentiate.

$$T = \frac{2}{9}\left(1200 - 3x + 5(x^2 + 90\,000)^{\frac{1}{2}}\right)$$

$\frac{2}{9}$ is a constant, differentiate using chain rule for $(x^2 + 90\,000)^{\frac{1}{2}}$.

$$\frac{dT}{dx} = \frac{2}{9}\left(-3 + \frac{5}{2}(x^2 + 90\,000)^{-\frac{1}{2}} \times 2x\right)$$

$$\frac{dT}{dx} = \frac{2}{9}\left(\frac{5x}{\sqrt{x^2 + 90\,000}} - 3\right)$$

for correct differentiation of $(x^2 + 90\,000)^{\frac{1}{2}}$ [1]

all correct [1]

We are asked to find the minimum value, this is when $\frac{dT}{dx} = 0$.

$$\frac{2}{9}\left(\frac{5x}{\sqrt{x^2+90\,000}}-3\right)=0$$

[1]

Rearrange and solve for x.

$$\frac{5x}{\sqrt{x^2+90\,000}}=3$$

$$5x=3\sqrt{x^2+90\,000}$$

Square both sides.

$$25x^2=9\left(x^2+90\,000\right)$$
$$16x^2=810\,000$$
$$x^2=50625$$
$$x=\pm 225$$

[1]

As x is a distance, it can only take a positive value.

$$x=225$$

[1]

Substitute $x=225$ into the equation for T.

$$T=\frac{2}{9}\left(1200-3\times 225+5\sqrt{225^2+90\,000}\right)$$
$$T=\frac{1600}{3}$$

The minimum value of T is $\frac{1600}{3}$ (seconds) [1]

(6 marks)

6 (a) A curve has the equation $y = e^{(4x - x^2)}$.

Find and factorise an expression for $\frac{dy}{dx}$.

Answer

Differentiate $y = e^{(4x - x^2)}$ using the chain rule

i.e. substitute $y = e^u$ and $u = 4x - x^2$ into $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = e^u \times (4 - 2x)$$

[M1]

Mark Scheme and Guidance

This mark is for attempting to use the chain rule.

Substitute back in $u = 4x - x^2$ to get the first derivative in x

$$\frac{dy}{dx} = (4 - 2x)e^{(4x - x^2)}$$

Factorise out 2 from the first bracket

$$\frac{dy}{dx} = 2(2 - x)e^{(4x - x^2)}$$

[A1]

(2 marks)

(b) Hence find $\int \frac{(2 - x)e^{4x}}{e^{x^2}} dx$.

Answer

The first derivative from part (a) says that if $y = e^{(4x - x^2)}$ then

$$\frac{dy}{dx} = 2(2 - x)e^{(4x - x^2)}$$

Reverse the process to go from the first derivative back to the equation of the curve

$$\int 2(2 - x)e^{(4x - x^2)}dx = e^{(4x - x^2)} + c$$

The integral in the question is almost the same as this, but needs rearranging first

$$\int \frac{(2 - x)e^{4x}}{e^{x^2}}dx = \int (2 - x)e^{4x - x^2}dx$$
$$= \frac{1}{2}\int 2(2 - x)e^{4x - x^2}dx$$

Now substitute the reversed integral from above into the right-hand side here

$$\int \frac{(2 - x)e^{4x}}{e^{x^2}}dx = \frac{1}{2}\left[e^{(4x - x^2)} + c\right]$$

Expand (you can rename the unknown constant $\frac{1}{2}c$ as c again)

$$\int \frac{(2 - x)e^{4x}}{e^{x^2}}dx = \frac{1}{2}e^{(4x - x^2)} + c$$

[B1 B1]

Mark Scheme and Guidance

B1: For at least $\frac{1}{2}e^{4x - x^2}$ (may not have $+c$).

B1: For $\frac{1}{2}e^{4x - x^2} + c$ (must have $+c$).

Examiner Tips and Tricks

Always add $+c$ to your answers to indefinite integrals, otherwise you will lose a mark!

(2 marks)

(c) Show that the second derivative of the curve $y = e^{(4x - x^2)}$ satisfies the relationship

$$\frac{d^2y}{dx^2} = (p + qx + rx^2)y$$

where p, q and r are constants to be found.

Answer

Differentiate $\frac{dy}{dx} = 2(2 - x)e^{(4x - x^2)}$ again to get $\frac{d^2y}{dx^2}$ but this time using the product rule $\frac{du}{dx}v + u\frac{dv}{dx}$

First find the derivatives of $u = 2(2 - x) = 4 - 2x$ and $v = e^{(4x - x^2)}$ (note that $\frac{dv}{dx}$ is the answer to part (a))

$$\frac{du}{dx} = -2$$

$$\frac{dv}{dx} = 2(2 - x)e^{(4x - x^2)}$$

Then substitute these into $\frac{d^2y}{dx^2} = \frac{du}{dx}v + u\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = (-2) \times e^{(4x - x^2)} + (4 - 2x) \times \left[2(2 - x)e^{(4x - x^2)}\right]$$

[M1]

Mark Scheme and Guidance

This mark is for attempting to use the product rule.

Simplify

$$\frac{d^2y}{dx^2} = -2e^{(4x-x^2)} + 4(2-x)(2-x)e^{(4x-x^2)}$$

[A1]

Mark Scheme and Guidance

This mark is for the correct result after using the product rule (it can be unsimplified and messy).

Factorise out $e^{(4x-x^2)}$ because the answer must be written in the form $(p+qx+rx^2)y$ and $y = e^{(4x-x^2)}$

$$\frac{d^2y}{dx^2} = [-2+4(2-x)(2-x)]e^{(4x-x^2)}$$

[M1]

Expand and simplify inside the brackets

$$\frac{d^2y}{dx^2} = (-2+4(4-4x+x^2))e^{(4x-x^2)}$$
$$= (-2+16-16x+4x^2)e^{(4x-x^2)}$$
$$= (14-16x+4x^2)e^{(4x-x^2)}$$

The result is meant to have the form $\frac{d^2y}{dx^2} = (p+qx+rx^2)y$

This is true, as $y = e^{(4x - x^2)}$

$$\frac{d^2y}{dx^2} = (14 - 16x + 4x^2)y$$

[A1]

(4 marks)