IGCSE · Cambridge (CIE) · Further Maths

2 hours · 19 questions

Exam Questions

Integration

Introduction to Integration / Integrating Powers of x / Definite Integrals / Finding Areas with Integration / Finding Areas Between Lines & Curves / Integrating Trig Functions / Integrating e^x & 1/x / Reverse Chain Rule

Medium (3 questions)	/18
Hard (11 questions)	/86
Very Hard (5 questions)	/44
Total Marks	**/148**

Medium Questions

1 (a) Giving your answer in its simplest form, find the exact value of

$$\int_0^4 \frac{10}{5x+2}\,dx$$

Answer

Take 10 out of the integral as a constant.

$$\int_0^4 \frac{10}{5x+2}\mathrm{d}x = 10\int_0^4 \frac{1}{5x+2}\mathrm{d}x$$

Integrate the fraction using reverse chain rule.

$$= 10\left[\frac{1}{5}\ln(5x+2)\right]_0^4$$

$k \times \ln(5x+2)$ [1]

Take $\frac{1}{5}$ as a factor.

$$= 10 \times \frac{1}{5}\left[\ln(5x+2)\right]_0^4$$

$$= 2\left[\ln(5x+2)\right]_0^4$$

[1]

Apply the limits.

$$= 2\left(\ln 22 - \ln 2\right)$$

[1]

$$= 2\ln\frac{22}{2}$$

$$= 2\ln 11$$

$2\ln 11$ [1]

(4 marks)

(b) $\int_0^{\ln 2} (e^{4x+2})^2 \, dx$

Answer

Use the law of logarithms, $(a^m)^n = a^{mn}$.

$$\int_0^{\ln 2} (e^{4x+2})^2 \, dx = \int_0^{\ln 2} (e^{2(4x+2)}) \, dx$$

$$= \int_0^{\ln 2} e^{8x+4} \, dx$$

[1]

Integrate.

$$= \left[\frac{1}{8} e^{8x+4} \right]_0^{\ln 2}$$

[1]

Apply the limits.

$$= \frac{1}{8} e^{8\ln 2 + 4} - \frac{1}{8} e^4$$

Factorise $\frac{1}{8}$.

$$= \frac{1}{8}(e^{8\ln 2 + 4} - e^4)$$

[1]

Rewrite the first exponential term as a product of 2 terms.

$$= \frac{1}{8}\left(e^{8 \ln 2} \times e^4 - e^4\right)$$

$$= \frac{1}{8}\left(e^{\ln 2^8} \times e^4 - e^4\right)$$

[1]

The exponential and natural log are inverses so cancel each other out.

$$= \frac{1}{8}\left(256e^4 - e^4\right)$$

Factorise e^4.

$$= \frac{1}{8}e^4(256 - 1)$$

$$= \frac{255}{8}e^4$$

$\frac{255}{8}e^4$ [1]

(5 marks)

2 Find $\frac{d}{dx}(16 - x^2)^{\frac{3}{2}}$ and hence evaluate the area enclosed by the curve $y = x\sqrt{16 - x^2}$ and the lines $y = 0$, $x = 1$ and $x = 3$.

Answer

Differentiate using the chain rule.

$$\frac{d}{dx}(16 - x^2)^{\frac{3}{2}} = \frac{3}{2}(16 - x^2)^{\frac{1}{2}} \times (-2x)$$

$$= -3x(16 - x^2)^{\frac{1}{2}}$$

attempt at chain rule [1]
fully correct [1]

The area required is given by the integral

$$\int_1^3 x(16 - x^2)^{\frac{1}{2}}\,dx$$

We have just found that the differential of $(16 - x^2)^{\frac{3}{2}}$ is $-3x(16 - x^2)^{\frac{1}{2}}$. This is almost the integrand for the area required – but is "-3" times bigger, so divide the integrand by -3.

$$\int_1^3 x(16 - x^2)^{\frac{1}{2}}\,dx = -\frac{1}{3}\int_1^3 x(16 - x^2)^{\frac{1}{2}}\,dx$$

$$= -\frac{1}{3}\left[(16 - x^2)^{\frac{3}{2}}\right]_1^3$$

for $-\frac{1}{3}$ [1]

for $k(16 - x^2)^{\frac{3}{2}}$ [1]

Apply the limits.

$$\left(-\frac{1}{3}(16 - 3^2)^{\frac{3}{2}}\right) - \left(-\frac{1}{3}(16 - 1^2)^{\frac{3}{2}}\right)$$

Area is 13.2 **units**^{2} [1]
(5 marks)

3 Find $\int \frac{1}{(7x+4)^m}\,dx$ in the following cases.

(a) $m = 2$

(b) $m = 1$

Answer

(a)

Substitute in $m = 2$

$$\int \frac{1}{(7x+4)^2}\,dx$$

Write it in the form $(ax+b)^n$ using negative indices

$$\int (7x+4)^{-2}\,dx$$

Use the rule $\int (ax+b)^n\,dx = \frac{1}{a} \times \frac{1}{n+1}(ax+b)^{n+1} + c$

$$\int (7x+4)^{-2}\,dx = \frac{1}{7} \times \frac{1}{-2+1}(7x+4)^{-2+1} + c$$

Simplify

$$= \frac{1}{7} \times \frac{1}{-1}(7x+4)^{-1} + c$$

$$= -\frac{1}{7}(7x+4)^{-1} + c$$

Present the final answer (including $+c$)

$$\int (7x+4)^{-2}\,dx = -\frac{1}{7}(7x+4)^{-1} + c$$

[B2]

Mark Scheme and Guidance

B1: For at least $-\frac{1}{7}(7x+4)^{-1}$ (may not have $+c$).

B1: $-\frac{1}{7}(7x+4)^{-1}+c$ (must have $+c$).

(b)

Substitute in $m=1$

$$\int \frac{1}{7x+4}\,\mathrm{d}x$$

Use the rule $\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$

$\int \frac{1}{7x+4}\,\mathrm{d}x = \frac{1}{7}\ln|7x+4| + c$

[B2]

Mark Scheme and Guidance

B1: For at least $\frac{1}{7}\ln|7x+4|$ (may not have $+c$).

B2: For $\frac{1}{7}\ln|7x+4| + c$ (must have $+c$).

(4 marks)

Hard Questions

1 Find the exact value of $\int_2^4 \frac{(x+1)^2}{x^2} \mathrm{d}x$.

Answer

Expand the numerator.

$$\frac{(x+1)^2}{x^2} = \frac{x^2+x+x+1}{x^2}$$

[1]

Simplify the numerator.

$$\frac{(x+1)^2}{x^2} = \frac{x^2+2x+1}{x^2}$$

The single term on the denominator means this can be rewritten as three separate terms.

$$\frac{(x+1)^2}{x^2} = \frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}$$

Simplify each of these terms.

$$\frac{(x+1)^2}{x^2} = 1 + \frac{2}{x} + \frac{1}{x^2}$$

[1]

Rewrite the integral and integrate using the usual rules for powers and $\frac{1}{x}$.

$$\int_2^4 (1+2x^{-1}+x^{-2})\,\mathrm{d}x = \left[\, x + 2\ln x - x^{-1} \,\right]_2^4$$

two terms correct term [1]

fully correct [1]

Apply the limits.

$$\left[x+2\ln x-x^{-1}\right]_2^4=\left(4+2\ln 4-\frac{1}{4}\right)-\left(2+2\ln 2-\frac{1}{2}\right)$$

[1]

Simplify and rewrite $2\ln 4$ as $2\ln 2^2$.

$$=\frac{15}{4}+2\ln\left(2^2\right)-\frac{3}{2}-2\ln 2$$

Simplify using the laws of logarithms.

$$=\frac{9}{4}+4\ln 2-2\ln 2$$

$$=\frac{9}{4}+2\ln 2$$

$$\int_2^4\left(\frac{(x+1)^2}{x^2}\right)dx=\frac{9}{4}+2\ln 2$$ [1]

(6 marks)

2 Given

$$\frac{d}{dx}(x\cos x)=-x\sin x+\cos x$$

find the exact value of $\int_0^{\frac{\pi}{6}} x\sin x\,dx$.

Answer

From the given information,

$$\int(-x\sin x+\cos x)\,dx=x\cos x+C$$

[1]

Therefore,

$$\int (x \sin x - \cos x)\,\mathrm{d}x = -x \cos x + C$$

Since the question is only asking us to integrate $x \sin x$, we need to account for the $-\cos x$. As we can integrate term by term we can negate the $-\cos x$ by adding the integral of $\cos x$.

$$\int \cos x = \sin x$$

[1]

Therefore,

$$\int (x \sin x)\,\mathrm{d}x = -x \cos x + C + \sin x$$

[1]

Apply the limits (and so ignore "+C").

$$[-x \cos x + \sin x]_0^{\pi/6} = \left[-\frac{\pi}{6}\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\right] - [0]$$

[1]

Simplify.

$$\left(-\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}$$

$\frac{1}{2} - \frac{\pi\sqrt{3}}{12}$ [1]

(5 marks)

3 (a) Given that $\int_1^a \left(\frac{2}{2x+3} + \frac{3}{3x-1} - \frac{1}{x}\right)\mathrm{d}x = \ln 2.4$ where $a > 1$, find the value of a.

Answer

First find the definite integral $\int_1^a \left(\frac{2}{2x+3} + \frac{3}{3x-1} - \frac{1}{x}\right) \mathrm{d}x$. Remember that $\int \left(\frac{a}{ax+b}\right) \mathrm{d}x = \ln(ax+b) + C$ (though we don't need C, the constant of integration, here as it's definite integration).

$$\int_1^a \left(\frac{2}{2x+3} + \frac{3}{3x-1} - \frac{1}{x}\right) \mathrm{d}x = \left[\ln(2x+3) + \ln(3x-1) - \ln x\right]_1^a$$

one correct term [1]
all terms correct [1]

Substitute the limits of integration.

$$= \ln(2a+3) + \ln(3a-1) - \ln a - (\ln(2+3) + \ln(3-1) - \ln 1)$$

[1]

Remember that "ln1" is 0.

$$= \ln(2a+3) + \ln(3a-1) - \ln a - (\ln 5 + \ln 2)$$

Apply the laws of indices.

$$= \ln(2a+3)(3a-1) - \ln a - \ln 10$$
$$= \ln(2a+3)(3a-1) - (\ln a + \ln 10)$$
$$= \ln(2a+3)(3a-1) - \ln(10a)$$
$$= \ln\left(\frac{(2a+3)(3a-1)}{10a}\right)$$

Now equate this to ln 2.4, as we are told in the question that $\int_1^a \left(\frac{2}{2x+3} + \frac{3}{3x-1} - \frac{1}{x}\right) \mathrm{d}x = \ln 2.4$

$$\ln\left(\frac{(2a+3)(3a-1)}{10a}\right) = \ln 2.4$$

[1]

We can now equate the arguments, or "remove the ln's".

$$\frac{(2a+3)(3a-1)}{10a}=2.4$$

Rearrange to a quadratic equation in the form $ax^2 + bx + c = 0$.

$$(2a+3)(3a-1)=24a$$
$$6a^2+7a-3=24a$$
$$6a^2-17a-3=0$$

[1]

Solve by factorising or using the quadratic formula.

$$6a^2-17a-3=0$$
$$(a-3)(6a+1)=0$$

$$a=3,\ a=-\frac{1}{6}$$

any one solution [1]

Remember that $a>1$ therefore discount $a=-\frac{1}{6}$ as a solution.

$a=3$

only $a=3$ [1]

(7 marks)

(b) (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}(6\sin^3 kx)$, where k is a constant.

(ii) Hence find $\int(\sin^2 2x\cos 2x)dx$.

Answer

i) We can differentiate $6\sin^3 kx$ using the chain rule, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$. First define *y* in terms of *u* and *u* in terms of *x*.

$$y=6u^3 \quad , \quad u=\sin kx$$

Now differentiate y with respect to u and u with respect to x . (It's helpful to know that $\frac{d}{dx}(\sin kx) = k\cos kx$ but if you need to, differentiate $\sin kx$ separately using the chain rule again to get this result.)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 18u^2 \qquad , \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = k\cos kx$$

Multiply the derivatives.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 18u^2 \cdot k\cos kx$$

And replace u with $\sin kx$.

$$= 18\sin^2 kx \cdot k\cos kx$$

Simplify to the final answer.

$$\mathbf{\frac{dy}{dx} = 18k\sin^2 kx\cos kx} \quad [2]$$

one mark for any multiply of k multiplied by $\sin^2 kx\cos kx$ and both marks for fully correct answer

ii) "Hence" and the fact that the integrand looks suspiciously like the answer to part (i) tells us that this is a problem to be solved using Reverse Chain Rule.

$$\int(\sin^2 2x \cos 2x)\mathrm{d}x = \int(\sin^2 kx \cos kx)\mathrm{d}x \text{ where } k = 2$$

However we need $\int(18k \sin^2 kx \cos kx)\mathrm{d}x$. Therefore, as $k = 2$, $\int(36 \sin^2 kx \cos kx)\mathrm{d}x$.

$$\int(\sin^2 2x \cos 2x)\mathrm{d}x = \frac{1}{36}\int(36 \sin^2 2x \cos 2x)\mathrm{d}x$$

$$= \frac{1}{36}\int(18k \sin^2 kx \cos kx)\mathrm{d}x \quad \text{where } k = 2$$

Therefore,

$$\int(\sin^2 2x\cos 2x)\mathrm{d}x = \frac{1}{36}(6\sin^3 kx + c)$$

Simplify for the final answer.

$\frac{1}{6}\sin^3 kx + c$ [2]

one mark for "$\frac{1}{6}\sin^3 kx$" and both marks if "$+\ c$" is included

Note that we don't write "$+\ \frac{1}{36}c$" as c is just some unknown constant

(4 marks)

4 Find $\int_3^5\left(\frac{1}{x-1} - \frac{1}{(x-1)^2}\right)\mathrm{d}x$, giving your answer in the form $a + \ln b$, where a and b are rational numbers.

Answer

Start by integrating each fraction separately.

To integrate $\frac{1}{x-1}$, use the result $\int\frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

$$\int\frac{1}{x-1}\mathrm{d}x = \ln|x-1| + C$$

For $\frac{1}{(x-1)^2} = (x-1)^{-2}$, use reverse chain rule.

$$\int(x-1)^{-2}\,\mathrm{d}x = \frac{1}{-1}(x-1)^{-1} + C$$

$$= -\frac{1}{x-1} + C$$

This means that

$$\int\left(\frac{1}{x-1}-\frac{1}{(x-1)^2}\right)dx = \ln|x-1| - -\frac{1}{x-1}+C$$

$$= \ln|x-1| + \frac{1}{x-1}+C$$

Now apply the limits (and so no need for "+C".)

$$\left[\ln|x-1| + \frac{1}{x-1}\right]_3^5$$

$\ln|x-1|$ [1]

$\frac{1}{x-1}$ [1]

Substitute $x=5$ into the function.

$$\ln|5-1| + \frac{1}{5-1} = \ln 4 + \frac{1}{4}$$

Substitute $x=3$ into the function.

$$\ln|3-1| + \frac{1}{3-1} = \ln 2 + \frac{1}{2}$$

Subtract the bottom limit value from the top limit value.

$$\left(\ln 4 + \frac{1}{4}\right) - \left(\ln 2 + \frac{1}{2}\right)$$

[1]

Expand the brackets, being careful with the negative before the second bracket.

$$\ln 4 - \ln 2 + \frac{1}{4} - \frac{1}{2}$$

Use logarithm laws to simplify $\ln 4 - \ln 2$.

$$\ln\left(\frac{4}{2}\right) - \frac{1}{4}$$

Finally, simplify.

$$-\frac{1}{4} + \ln 2$$ [2]

(5 marks)

5 A curve is such that $\frac{d^2y}{dx^2} = \sin\left(6x - \frac{\pi}{2}\right)$. Given that $\frac{dy}{dx} = \frac{1}{2}$ at the point $\left(\frac{\pi}{4}, \frac{13\pi}{12}\right)$ on the curve, find the equation of the curve.

Answer

Integrate $\sin\left(6x - \frac{\pi}{2}\right)$ using reverse chain rule to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \int \sin\left(6x - \frac{\pi}{2}\right) dx$$

$$\frac{dy}{dx} = -\frac{\cos\left(6x - \frac{\pi}{2}\right)}{6} + C$$

[2]

Substitute $\frac{dy}{dx} = \frac{1}{2}$ and the given x-coordinate value.

$$\frac{1}{2} = -\frac{\cos\left(6\left(\frac{\pi}{4}\right) - \frac{\pi}{2}\right)}{6} + C$$

$$\frac{1}{2} = -\frac{\cos\left(\frac{6\pi}{4} - \frac{\pi}{2}\right)}{6} + C$$

[1]

Rearrange and solve to find C.

$$C = \frac{1}{3}$$

Substitute into the equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\cos\left(6x - \frac{\pi}{2}\right)}{6} + \frac{1}{3}$$

Integrate $\frac{dy}{dx}$ using reverse chain rule to find the equation of the curve.

$$y = \int \left(-\frac{\cos\left(6x - \frac{\pi}{2}\right)}{6} + \frac{1}{3} \right) dx$$

$$y = -\frac{\sin\left(6x - \frac{\pi}{2}\right)}{36} + \frac{1}{3}x + A$$

[2]

Substitute the given coordinate into the equation of the curve to work out A.

$$\frac{13\pi}{12} = -\frac{\sin\left(6\left(\frac{\pi}{4}\right) - \frac{\pi}{2}\right)}{36} + \frac{1}{3}\left(\frac{\pi}{4}\right) + A$$

[1]

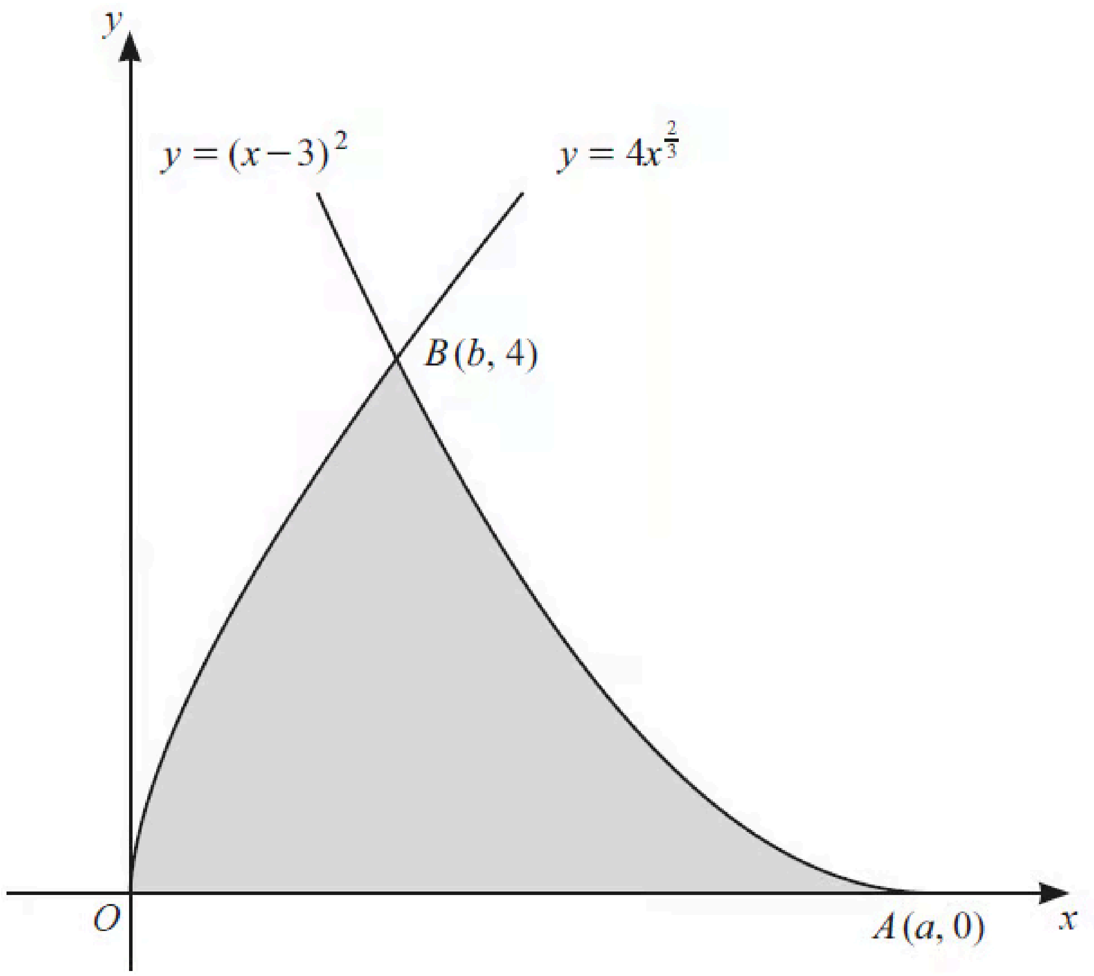
$$A = \pi$$

Substitute into the equation of the curve.

$$y = -\frac{1}{36}\sin\left(6x - \frac{\pi}{2}\right) + \frac{1}{3}x + \pi$$ [1]

(7 marks)

6 (a)



The diagram shows part of the graphs of $y=4x^{\frac{2}{3}}$ and $y = (x-3)^2$ .The graph of $y = (x-3)^2$ meets the x-axis at the point $A(a, 0)$ and the two graphs intersect at the point $B(b, 4)$.

Find the value of a and of b.

Answer

The root of the equation $y = (x - 3)^2$ is at $x = 3$.

$$\therefore a = 3$$

Substitute the y–coordinate of B into the equation $y=(x - 3)^2$.

$$4 = (x - 3)^2$$

Square root both sides.

$$\pm 2 = x - 3$$

$$x = 5 \text{ or } x = 1$$

Looking at the graph, $x = 5$ cannot be the x co-ordinate of B so disregard this solution.

$a = 3$ [1]
$b = 1$ [1]
(2 marks)

(b) Find the area of the shaded region.

Answer

To find the shaded area, integrate under each curve.

$$\int_0^1 4x^{\frac{2}{3}}\,dx + \int_1^3 (x - 3)^2 dx$$

[1]

Integrate the first part normally and use reverse chain rule on the second part.

$$\left[\frac{4x^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^1 + \left[\frac{(x - 3)^3}{3} \times \frac{1}{1}\right]_1^3$$

[2]

Apply the limits.

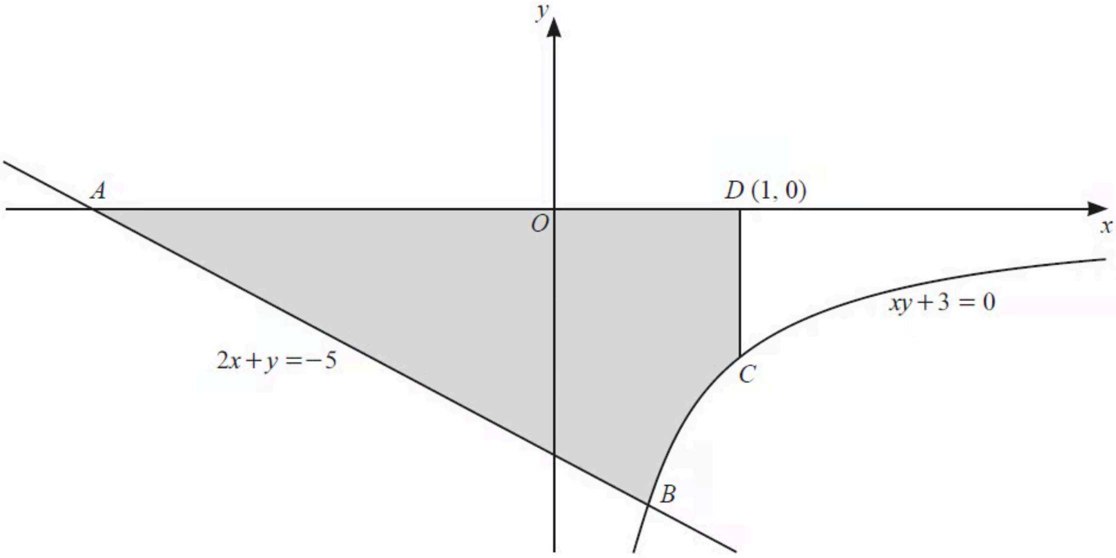
$$\left(\frac{12}{5} - 0\right) + \left(0 - -\frac{8}{3}\right)$$

[1]

$$\text{Shaded area} = \frac{76}{15} \text{ units}^2$$ [1]

(5 marks)

7 (a)



The diagram shows the straight line $2x + y = -5$ and part of the curve $xy + 3 = 0$. The straight line intersects the x-axis at the point A and intersects the curve at the point B. The point C lies on the curve. The point D has coordinates (1, 0). The line CD is parallel to the y-axis.

Find the coordinates of each of the points A and B.

Answer

Point A has y coordinate 0.
Substitute $y = 0$ into $2x + y = -5$ to find x.

$$2x + y = -5$$

$$2x + 0 = -5$$

$$x = -\frac{5}{2}$$

$A\left(-\frac{5}{2}, 0\right)$ [1]

Point B occurs when the straight line and curve intersect. Rearrange the equation of the straight line to make y the subject.

$$2x + y = -5$$

$$y = -5 - 2x$$

Substitute $y = -5 - 2x$ into the curve.

$$xy + 3 = 0$$

$$x(-5 - 2x) + 3 = 0$$

Expand the bracket.

$$-5x - 2x^2 + 3 = 0$$

Rearrange the equation so that the x^2 term is positive.

$$2x^2 + 5x - 3 = 0$$

Solve the quadratic equation using the quadratic formula.

$$x = \frac{1}{2}, -3$$

[1]

From the graph, x is positive when they intersect.

$$x = \frac{1}{2}$$

Find y by substituting $x = \frac{1}{2}$ into $y = -5 - 2x$.

$$y = -5 - 2\left(\frac{1}{2}\right)$$

$$y = -6$$

$B\left(\frac{1}{2}, -6\right)$ [1]

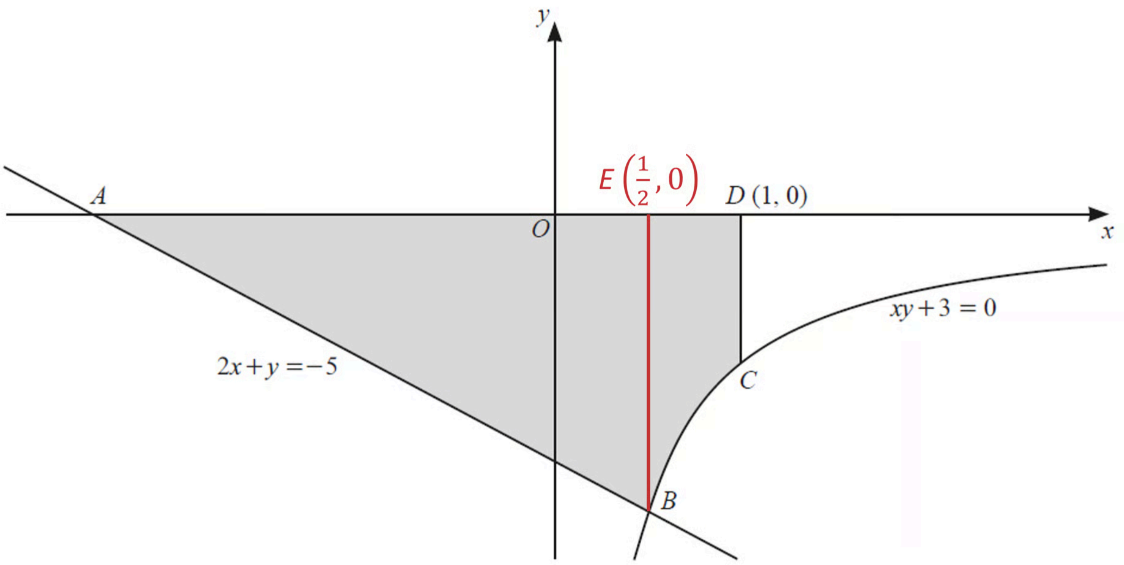
(3 marks)

(b) Find the area of the shaded region, giving your answer in the form $p+\ln q$, where p and q are positive integers.

Answer

Work out the area of the shaded area in 2 sections – a triangle and the area 'under' the curve.

Let point E be $\left(\frac{1}{2}, 0\right)$



Shaded area = area of triangle *ABE* + area 'under' the curve between *E* and *D*.

$$\text{Area of triangle} = \frac{1}{2}\times\left(\frac{5}{2}+\frac{1}{2}\right)\times 6$$
$$=9$$

[1]

Find the area of the remaining section using integration. Rearrange the curve to make y the subject.

$$xy+3=0$$

$$y=-\frac{3}{x}$$

Integrate with respect to x from $x=\frac{1}{2}$ to $x=1$.

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = \left[\ -3\ln x\ \right]_{\frac{1}{2}}^{1}$$

[1]

Apply the limits.

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = (-3\ln 1) - \left(-3\ln\left(\frac{1}{2}\right)\right)$$

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = 0 + 3\ln\frac{1}{2}$$

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = 3\ln\left(\frac{1}{2}\right)$$

[1]

Rewrite $\frac{1}{2}$ as 2^{-1}.

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = 3\ln(2^{-1})$$

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\mathrm{d}x = -3\ln 2$$

[1]

The answer is negative because the area is below the x-axis so remember to use the positive area to work out the total area of the shaded region.

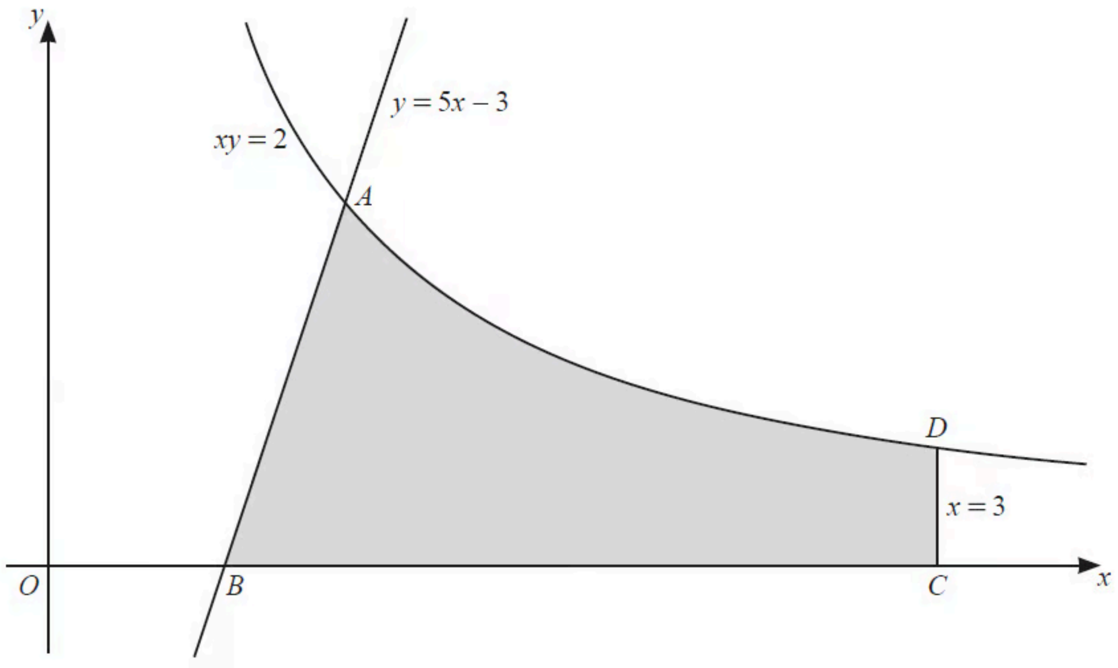
realisation that value of integral is negative and making the adjustment [1]

$$\text{Total area} = 9 + 3\ln 2$$
$$\text{Total area} = 9 + \ln 2^3$$

$\text{Area} = 9 + \ln 8$ [1]

(6 marks)

8



The diagram shows part of the curve $xy = 2$ intersecting the straight line $y = 5x - 3$ at the point A.

The straight line meets the x-axis at the point B. The point C lies on the x-axis and the point D lies on the curve such that the line CD has equation $x = 3$. Find the exact area of the shaded region, giving your answer in the form $p + \ln q$, where p and q are constants.

Answer

Substitute $y = 5x - 3$ into $xy = 2$, rearrange and solve by factorisation to find the x coordinate where the two lines intersect.

$$x(5x-3) = 2$$
$$5x^2 - 3x - 2 = 0$$

$$(5x+2)(x-1) = 0$$

$$x = -\frac{2}{5} \text{ and } x = 1$$

for attempt at a three term quadratic equation with solution [1] *correct working and solution of* $x = 1$ [1]

The graph is in the first quadrant, therefore the intersection we need is when $x = 1$.

Find the y-coordinate when $x = 1$.

$$(1)(y) = 2$$
$$y = 2$$

Therefore,

$$\text{Point A} = (1, 2)$$

[1]

Point B is where the line $y = 5x - 3$ crosses the $x$axis.

$$0 = 5x - 3$$
$$x = \frac{3}{5}$$

Therefore,

$$\text{Point B} = \left(\frac{3}{5}, 0\right)$$

[1]

The shaded area can be split into a triangle and the area under the curve $xy = 2$ between $x = 1$ and $x = 3$.
Find the area of the triangle.

$$\text{Area} = \frac{(x \text{ coordinate of } A - x \text{ coordinate of } B) \times 2}{2}$$

$$\text{Area} = \left(1 - \frac{3}{5}\right)$$

$$\text{Area} = \frac{2}{5}$$

[1]

Integrate and apply limits to find the area under the curve between $x = 1$ and $x = 3$.

$$\int_1^3 \left(\frac{2}{x}\right) dx = [2\ln x]_1^3$$

[1]

$$= [2 \ln(3)] - [2\ln(1)]$$
$$= 2 \ln(3)$$

[1]

Add together the area under the curve and the area of the triangle for the total area.

$$\frac{2}{5} + 2 \ln(3)$$

Use the logarithm power law to rewrite in the required way.

$$\frac{2}{5} + \ln(3^2)$$

$\frac{2}{5} + \ln(9)$ [1]

(8 marks)

9 (a) (i) Given that $f(x) = \dfrac{1}{\cos x}$, show that $f'(x) = \tan x \sec x$.

[3]

(ii) Hence find $\int\left(3\tan x\sec x-\sqrt[4]{e^{3x}}\right)\mathrm{d}x$.

[3]

Answer

(i) $\frac{1}{\cos x}=(\cos x)^{-1}$.

Differentiate $y=(\cos x)^{-1}$ with respect to x using the chain rule.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-(\cos x)^{-2}\times\sin x$$

[2]

Simplify.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin x}{\cos x}\times\frac{1}{\cos x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tan x\sec x$$ [1]

(ii) Rewrite $\sqrt[4]{e^{3x}}$ as a power of e.

$$e^{\frac{3x}{4}}$$

[1]

Integrate using reverse chain rule.

$$\int e^{\frac{3x}{4}}=\frac{4}{3}e^{\frac{3x}{4}}+C$$

From part (i), we know that $\int(\tan x\sec x)\,\mathrm{d}x=\frac{1}{\cos x}+C$.

Therefore,

$$\int (3 \tan x \sec x)\, dx = \frac{3}{\cos x} + C$$

[1]

Combine to answer the original question.

$$\int \left(3 \tan x \sec x - \sqrt[4]{e^{3x}}\right) dx = \frac{3}{\cos x} - \frac{4}{3} e^{\frac{3x}{4}} + C$$ [1]

(6 marks)

(b) Given that $\int_2^5 \frac{p}{px+10} dx = \ln 2$, find the value of the positive constant p.

Answer

Integrate $\frac{p}{px+10}$ using reverse chain rule – it is of the form $\frac{f'(x)}{f(x)}$.

$$\int \frac{p}{px+10} dx = \ln(px+10) + C$$

Rewrite using the limits given in the question, and put equal to $\ln 2$.

$$[\ln(px+10)]_2^5 = \ln 2$$

[1]

Apply the limits on the left-hand side.

$$\ln(5p+10) - \ln(2p+10) = \ln 2$$

[1]

Use logarithm laws to rewrite.

$$\ln\left(\frac{5p+10}{2p+10}\right) = \ln 2$$

[1]

Therefore,

$$\frac{5p+10}{2p+10}=2$$

Rearrange and solve.

$$5p+10=2(2p+10)$$

[1]

$$5p-4p=20-10$$

$p=10$ [1]

(5 marks)

10 (a) Given that $\int_1^a \left(\frac{1}{x} - \frac{1}{2x+3}\right) dx = \ln 3$, where $a > 0$, find the exact value of a, giving your answer in simplest surd form.

Answer

Integrate the left-hand side of the equation.

$$\int_1^a \left(\frac{1}{x} - \frac{1}{2x + 3}\right) dx = \left[\ln x - \frac{1}{2}\ln(2x + 3)\right]_1^a$$

$\ln(x)$ [1]

$-\frac{1}{2}\ln(2x + 3)$ [1]

Apply the limits.

$$\left(\ln a - \frac{1}{2}\ln(2a + 3)\right) - \left(\ln(1) - \frac{1}{2}\ln(5)\right)$$

[1]

Expand the brackets and set equal to the right-hand side.

$$\ln a - \frac{1}{2}\ln(2a + 3) + \frac{1}{2}\ln(5) = \ln 3$$

Apply the law of logarithms, $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$$\ln a + \frac{1}{2}\ln\left(\frac{5}{2a + 3}\right) = \ln 3$$

Subtract $\ln a$ from both sides and apply the same law again.

$$\frac{1}{2}\ln\left(\frac{5}{2a + 3}\right) = \ln\left(\frac{3}{a}\right)$$

Apply the log law $a\ln b = \ln b^a$.

$$\ln\left[\left(\frac{5}{2a+3}\right)^{\frac{1}{2}}\right] = \ln\left(\frac{3}{a}\right)$$

[1]

Take exponentials of both sides.

$$\left(\frac{5}{2a+3}\right)^{\frac{1}{2}} = \frac{3}{a}$$

Square both sides.

$$\frac{5}{2a+3} = \frac{9}{a^2}$$

Multiply through by the denominators.

$$5a^2 = 9(2a+3)$$

Expand the brackets and then rearrange to make the equation equal to 0.

$$5a^2 - 18a - 27 = 0$$

[1]

Use the quadratic formula (and/or a calculator) with $a = 5$, $b = -18$, $c = -27$.
Disregard the negative solution since $a > 0$.

$$\boldsymbol{a = \frac{9 + 6\sqrt{6}}{5}}$$ [1]

(6 marks)

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} \left(\sin\left(2x + \frac{\pi}{3}\right) - 1 + \cos 2x\right) \mathrm{d}x$.

Answer

Integrate using reverse chain rule – i.e. $\sin(ax+b)$ integrates to $-\frac{1}{a}\cos(ax+b)$.

$$\left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)-x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)$ [1]

$+\frac{1}{2}\sin 2x$ [1]

$-x$ [1]

Apply the limits.

$$\left(-\frac{1}{2}\cos\left(2\left(\frac{\pi}{3}\right)+\frac{\pi}{3}\right)-\frac{\pi}{3}+\frac{1}{2}\sin 2\left(\frac{\pi}{3}\right)\right)-\left(-\frac{1}{2}\cos\left(\frac{\pi}{3}\right)\right)$$

[1]

Evaluate.

$$\frac{1}{2}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}+\frac{1}{4}$$

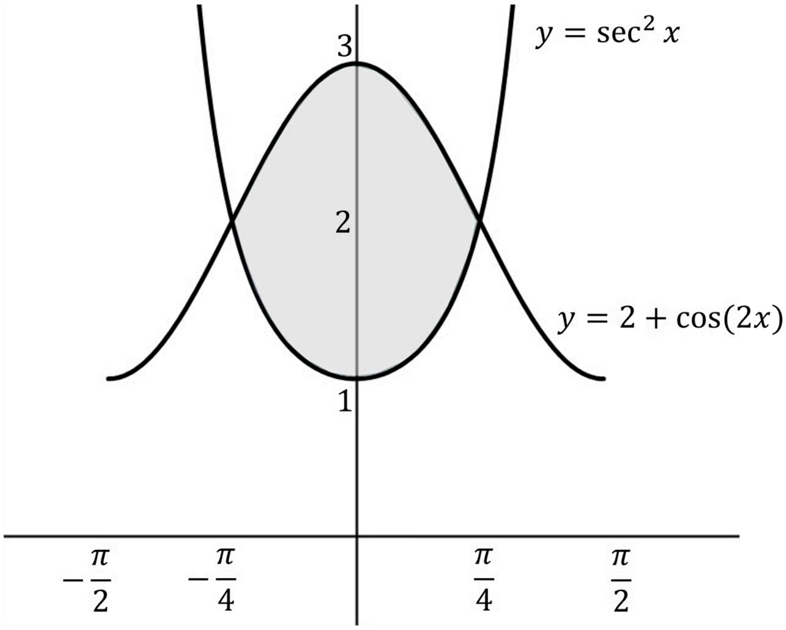
Simplify.

$\mathbf{\frac{3}{4}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}}$ [1]

(5 marks)

11 Throughout this question, x is measured in radians.

The curves $y = \sec^2 x$ and $y = 2 + \cos(2x)$ intersect at the points $\left(-\frac{\pi}{4}, 2\right)$ and $\left(\frac{\pi}{4}, 2\right)$, as shown.



Find the exact area of the shaded region enclosed.

Answer

The area enclosed between the top curve $y = \sec^2 x$ and the bottom curve $y = 2 + \cos(2x)$ between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ is

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sec^2 x - (2 + \cos(2x))\right) \mathrm{d}x$$

[M1 A1]

Mark Scheme and Guidance

M1: For either $\int \left(\sec^2 x - (2 + \cos(2x))\right) \mathrm{d}x$ or $\int \sec^2 x \, \mathrm{d}x - \int (2 + \cos(2x)) \, \mathrm{d}x$

.

A1: For either statement in M1 with the correct limits.

Expand and simplify inside the integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - (2 + \cos(2x)))\ \mathrm{d}x$$

Integrate the first term by reversing the fact that $\tan x$ differentiates to $\sec^2 x$

Integrate the last term using $\int \cos(ax)\mathrm{d}x = \frac{1}{a}\sin(ax) + c$

$$\tan x - 2x - \frac{1}{2}\sin 2x$$

[M1 A1]

Mark Scheme and Guidance

M1: For seeing either $\tan x$ or $k\sin 2x$.

A1: For all 3 terms integrated correctly.

Substitute in the limits

$$\left[\tan x - 2x - \frac{1}{2}\sin 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \left(\tan\frac{\pi}{4} - 2(\frac{\pi}{4}) - \frac{1}{2}\sin\left(\frac{2\pi}{4}\right)\right) - \left(\tan\left(-\frac{\pi}{4}\right) - 2(-\frac{\pi}{4}) - \frac{1}{2}\sin\left(-\frac{2\pi}{4}\right)\right)$$

[M1]

Simplify using your calculator

$$= \left(1 - \frac{\pi}{2} - \frac{1}{2}\right) - \left(-1 + \frac{\pi}{2} + \frac{1}{2}\right)$$
$$= 1 - \pi$$

$(1 - \pi)$ **square units**

[A1]

(6 marks)

Very Hard Questions

1 (a) Show that $\frac{1}{x+1} + \frac{2}{3x+10}$ can be written as $\frac{5x+12}{3x^2+13x+10}$

Answer

Write the two fractions as one with a single denominator of $(x+1)(3x+10)$.

$$\frac{1(3x+10)+2(x+1)}{(x+1)(3x+10)}$$

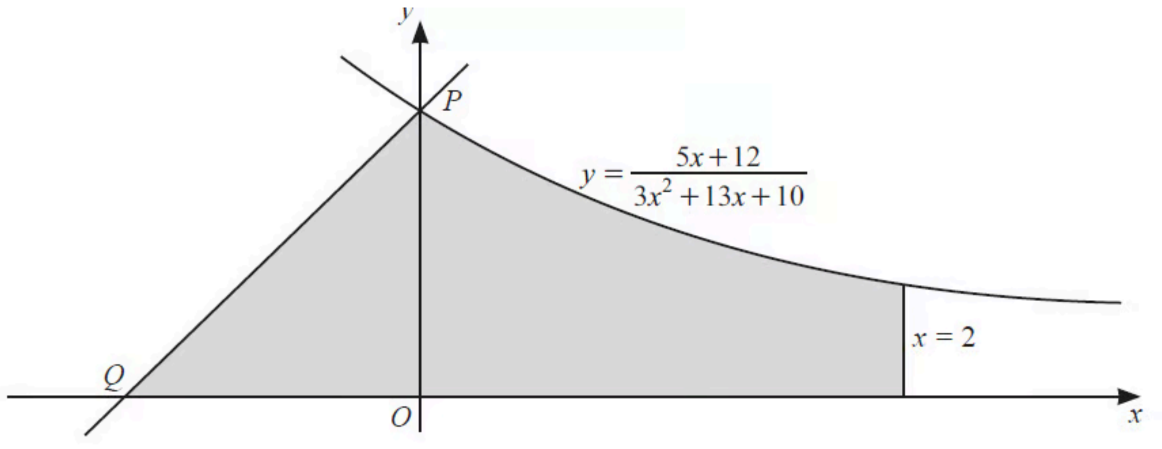
Expand and simplify.

$$= \frac{(3x+10)+2(x+1)}{(x+1)(3x+10)}$$
$$= \frac{3x+2x+10+2}{3x^2+10x+3x+10}$$
$$= \frac{5x+12}{3x^2+13x+10}$$

[1]

(1 mark)

(b)



The diagram shows part of the curve $y = \frac{5x + 12}{3x^2 + 13x + 10}$, the line $x = 2$ and a straight line of gradient 1. The curve intersects the y-axis at the point P. The line of gradient 1 passes through P and intersects the x-axis at the point Q. Find the area of the shaded region, giving your answer in the form $a + \frac{2}{3}\ln\left(b\sqrt{3}\right)$, where a and b are constants.

Answer

Find the y-intercept of the line, point *P*, by substituting $x = 0$ into the equation of the curve.

$$y = \frac{5(0) + 12}{3(0)^2 + 13(0) + 10}$$

$$y = \frac{12}{10}$$

$$y = \frac{6}{5}$$

Therefore,

$$P\left(0, \frac{6}{5}\right)$$

The line segment *PQ* has a gradient of 1, and a y-intercept of $\frac{6}{5}$. Substitute these values into the equation of line, $y = mx + c$, to find the equation of the line.

$$y = x + \frac{6}{5}$$

The line crosses the x-axis when $y = 0$.

$$0 = x + \frac{6}{5}$$

$$x = -\frac{6}{5}$$

Therefore,

$$Q\left(-\frac{6}{5}, 0\right)$$

for both points correct [1]

Find the area of the triangle *PQO*.

$$\text{Area triangle} = \frac{1}{2} \times \frac{6}{5} \times \frac{6}{5}$$

$$= \frac{18}{25}$$

[1]

Integrate the curve between 0 and 2 to find the area under the curve, using the (reverse of the) result in part (a).

$$\int_0^2 \left(\frac{1}{x+1} + \frac{2}{3x+10}\right) \mathrm{d}x$$

using part (a) [1]

$$\int_0^2 \left(\frac{1}{x+1} + \frac{2}{3x+10}\right) \mathrm{d}x = \left[\ln(x+1) + \frac{2}{3}\ln(3x+10)\right]_0^2$$

one for each term [2]

Apply the limits.

$$\left[\ln(3)+\frac{2}{3}\ln(16)\right]-\left[\ln(1)+\frac{2}{3}\ln(10)\right]=\ln(3)+\frac{2}{3}\ln(16)-\frac{2}{3}\ln(10)$$

[1]

Use the laws of logarithms to rewrite the first term, creating a factor of $\frac{2}{3}$ in every term, and then to combine all three terms as a single logarithm.

$$\frac{2}{3}\ln(3)^{\frac{3}{2}}+\frac{2}{3}\ln(16)-\frac{2}{3}\ln(10)=\frac{2}{3}\ln\left(3\sqrt{3}\right)+\frac{2}{3}\ln(16)-\frac{2}{3}\ln(10)$$
$$=\frac{2}{3}\left(\ln\left(\frac{48\sqrt{3}}{10}\right)\right)$$

rewriting the first term [1]
simplify to a single term [1]

Simplify so this part of the shaded area is in the correct form.

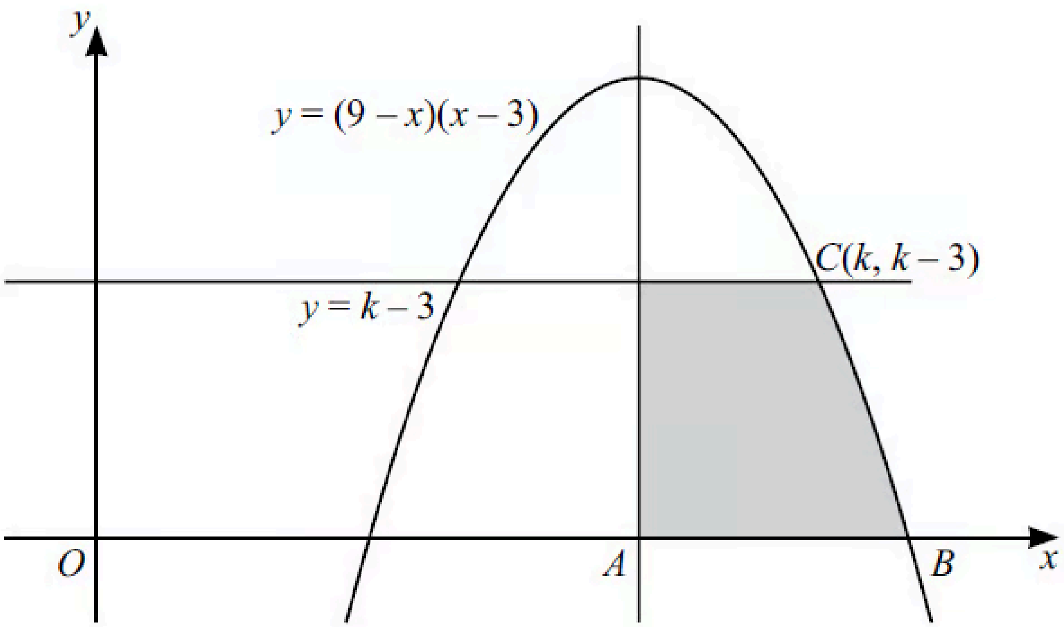
$$\frac{2}{3}\ln\left(\frac{24\sqrt{3}}{5}\right)$$

Add together the area of the triangle and the area under the curve.

$$\mathbf{\frac{18}{25}+\frac{2}{3}\ln\left(\frac{24\sqrt{3}}{5}\right)}$$ [1]

(9 marks)

2



The diagram shows part of the curve $y = (9 - x)(x - 3)$ and the line $y = k - 3$, where $k > 3$.

The line through the maximum point of the curve, parallel to the y-axis, meets the x-axis at A.

The curve meets the x-axis at B, and the line $y = k - 3$ meets the curve at the point $C(k, k - 3)$.

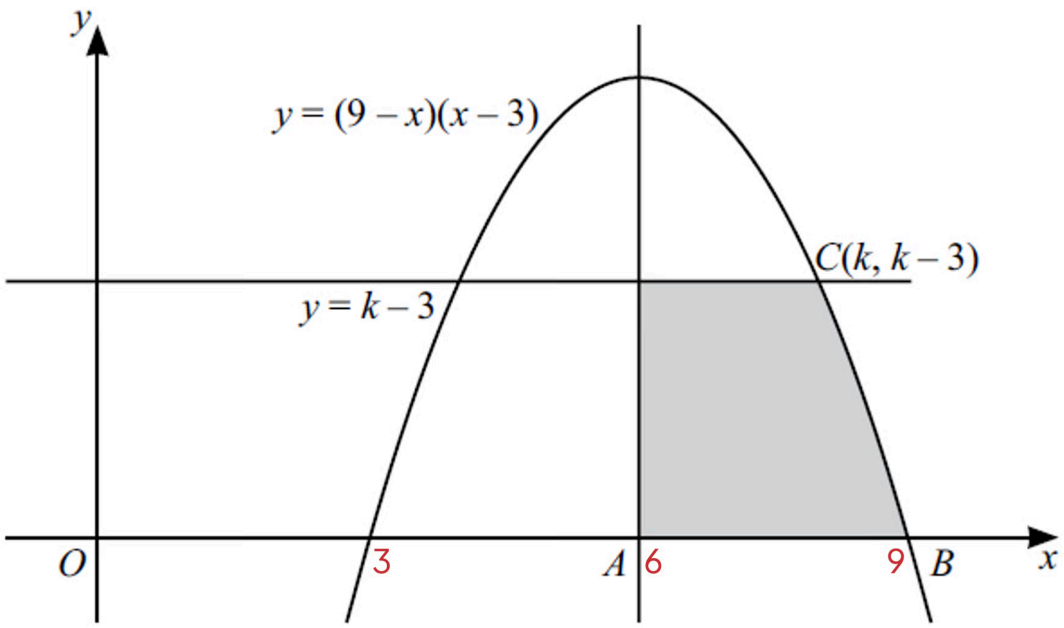
Find the area of the shaded region.

Answer

Using $y=0$ we can find the x-intercepts of the parabola.

$$(9-x)(x-3)=0$$
$$x=9\,,\ \ x=3$$

And using the symmetry of the curve we see that the *x*-coordinate of *A* is 6. (We could also find this by rewriting y as $y = -x^2 + 12x - 27$ and using the axis of symmetry, $x = \frac{-b}{2a}$.)



B = *9* [1]
A = *6* [1]

Now to find k, substitute the point $(k, k-3)$ into the equation for y.

$$k - 3 = (9 - k)(k - 3)$$

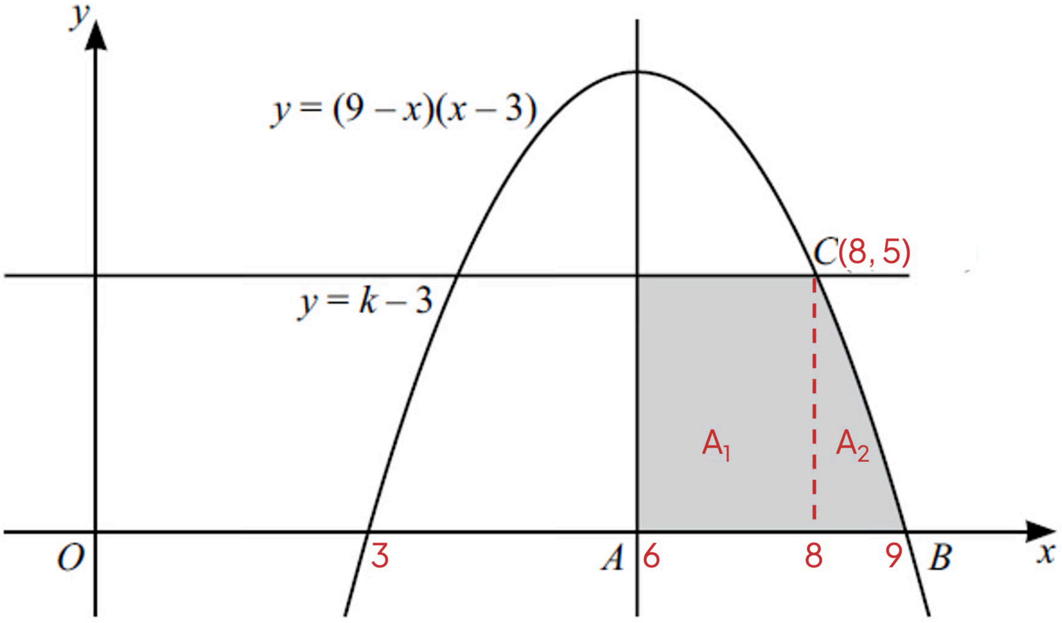
[1]

And solve for *k*.

$$k - 3 = 9k - 27 - k^2 + 3k$$
$$k^2 - 11k + 24 = 0$$
$$(k - 3)(k - 8) = 0$$
$$k \neq 3,\ k = 8$$

(These are the two *x*-values where $y = (9 - x)(x - 3)$ intersects $y = k - 3$ but *C* is the larger (furthest right) of the two so we know $k \neq 3$.)

k = 8 [1]



Adding this information to the diagram, we see that we can split the shaded area into two areas; A_1 which is a rectangle with width 2 (from 8 − 6) and height 5, and A_2.

$$A_1 = 2 \times 5 = 10$$

[1]

To find A_2, find the definite integral of $y = (9 - x)(x - 3)$ between 8 and 9.

$$\begin{aligned} A_2 &= \int_8^9 ((9 - x)(x - 3))\,dx \\ &= \int_8^9 (-x^2 + 12x - 27)\,dx \\ &= \left[\frac{-x^3}{3} + \frac{12x^2}{2} - 27x\right]_8^9 \\ &= \left[\frac{-x^3}{3} + 6x^2 - 27x\right]_8^9 \end{aligned}$$

one mark for two correct terms and two marks for all three terms correct [2]

Therefore the total shaded area is

$$A_1 + A_2 = 10 + \left(-\frac{9^3}{3} + 6(9)^2 - 27(9)\right) - \left(-\frac{8^3}{3} + 6(8)^2 - 27(8)\right)$$

[1]

$$= 10 + 0 - \left(-2\frac{2}{3}\right)$$

$$= 10 + 2\frac{2}{3}$$

Area = $12\frac{2}{3}$ units^2 [1]

(9 marks)

3 (a) Show that $\frac{3}{2x-3} + \frac{3}{2x+3}$ can be written as $\frac{12x}{4x^2-9}$.

Answer

To add together two fractions, we need a common denominator.
Find the lowest common denominator by multiplying $2x - 3$ by $2x + 3$.

lowest common denominator is $(2x + 3)(2x - 3)$

Write each fraction with a common denominator.

$$\frac{3}{2x - 3} + \frac{3}{2x + 3} = \frac{3(2x + 3)}{(2x - 3)(2x + 3)} + \frac{3(2x - 3)}{(2x - 3)(2x + 3)}$$

Write as a single fraction.

$$\frac{3(2x + 3) + 3(2x - 3)}{(2x - 3)(2x + 3)}$$

[1]

Expand the brackets.

$$\frac{6x + 9 + 6x - 9}{4x^2 + 6x - 6x - 9}$$

Collecting like terms.

$$\frac{12x}{4x^2 - 9}$$ [1]

(2 marks)

(b) Hence find $\int \frac{12x}{4x^2 - 9} \mathrm{d}x$, giving your answer as a single logarithm and an arbitrary constant.

Answer

Using part (a) to write the integral as two separate fractions.

$$\int \frac{3}{2x - 3} + \frac{3}{2x + 3} \mathrm{d}x$$

1 mark for each correct term [1]

Use the result $\int \frac{f'(x)}{f(x)} \mathrm{d}x = \ln|f(x)| + c$.

$$\frac{3}{2}\ln(2x - 3) + \frac{3}{2}\ln(2x + 3) + c$$

Use the logarithm law $\ln(m) + \ln(n) = \ln(m \times n)$.

$$\frac{3}{2}\ln(2x - 3)(2x + 3) + c$$

$$\frac{3}{2}\ln(4x^2 - 9) + c$$ [1]

(3 marks)

(c) Given that $\int_2^a \frac{12x}{4x^2 - 9} \mathrm{d}x = \ln 5\sqrt{5}$, where $a > 2$, find the exact value of a.

Answer

Using the answer to part (b).

$$\int_2^a \frac{12}{4x^2 - 9}\,dx = \left[\frac{3}{2}\ln(4x^2 - 9)\right]_2^a$$

Apply the limits.

$$\frac{3}{2}\ln(4a^2 - 9) - \frac{3}{2}\ln(4(2)^2 - 9) = \frac{3}{2}\ln(4a^2 - 9) - \frac{3}{2}\ln(7)$$

The right hand side of the equation, $\ln 5\sqrt{5}$ can be written as $\ln 5^{\frac{3}{2}}$.

Use the logarithm law $\ln(a)^m = m\ln(a)$.

$$\frac{3}{2}\ln(4a^2 - 9) - \frac{3}{2}\ln(7) = \frac{3}{2}\ln(5)$$

[1]

Collect like terms and divide each term by $\frac{3}{2}$.

$$\ln(4a^2 - 9) = \ln(5) + \ln(7)$$

Using the logarithm law $\ln(m) + \ln(n) = \ln(m \times n)$.

$$\ln(4a^2 - 9) = \ln(35)$$

Taking exponentials of both sides will cancel out the logarithm.

$$4a^2 - 9 = 35$$

[1]

$$4a^2 = 44$$

$$a^2 = 11$$

The question states that $a > 2$, therefore,

$a = \sqrt{11}$ [2]

(4 marks)

4 A curve is such that $\frac{d^2y}{dx^2} = 5\cos 2x$. This curve has a gradient of $\frac{3}{4}$ at the point $\left(-\frac{\pi}{12}, \frac{5\pi}{4}\right)$. Find the equation of this curve.

Answer

Integrate both sides of the equation.

$$\frac{dy}{dx} = \int 5\cos 2x \, dx = \frac{5}{2}\sin 2x + c$$

$k\sin 2x$ [1]

fully correct integration [1]

The gradient is given as $\frac{3}{4}$, so,

$$\frac{3}{4} = \frac{5}{2}\sin 2x + c$$

Divide each term by $\frac{5}{2}$.

$$\frac{3}{10} = \sin 2x + \frac{2}{5}c$$

Substitute in $x = -\frac{\pi}{12}$.

$$\frac{3}{10} = -\frac{1}{2} + \frac{2}{5}c$$

[1]

$$\frac{4}{5} = -\frac{2}{5}c$$

$$c = 2$$

[1]

Substitute this value of c into the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{5}{2}\sin 2x + 2$$

Integrate both sides again.

$$y = -\frac{5}{4}\cos 2x + 2x + d$$

any attempt to integrate [1]

fully correct integration [1]

Substitute in $y = \frac{5\pi}{4}$ and $x = -\frac{\pi}{12}$.

$$\frac{5\pi}{4} = -\frac{5}{4}\cos\left(2\left(-\frac{\pi}{12}\right)\right) + 2\left(-\frac{\pi}{12}\right) + d$$

[1]

$$\frac{5\pi}{4} = -\frac{5\sqrt{3}}{8} - \frac{\pi}{6} + d$$

Collect like terms.

$$d = \frac{17\pi}{12} + \frac{5\sqrt{3}}{8}$$

Substitute this into the expression for y.

$$y = -\frac{5}{4}\cos 2x + 2x + \frac{17\pi}{12} + \frac{5\sqrt{3}}{8}$$ [1]

(8 marks)

5 (a) The gradient of the normal to a curve at the point (x, y) is given by $\frac{x}{x+1}$

Given that the curve passes through the point $(1, 4)$, show that its equation is $y = 5 - \ln x - x$.

Answer

The gradient of the tangent is the negative reciprocal of the gradient of the normal, thereffore,

$$\frac{dy}{dx} = -\frac{(x+1)}{x} = -1 - \frac{1}{x}$$

using negative reciprocal [1]
correct derivative [1]

Integrate to find y.

$$y = \int\left(-1 - \frac{1}{x}\right)dx$$

$$y = -x - \ln x + c$$

$\ln x$ [1]
all part correct and + c [1]

Substitute the coordinates of the given point.

$$4 = -1 - \ln 1 + c$$

$$5 = c$$

Now we can write the equation of the curve.

$$y = -x - \ln x + 5$$

$y = 5 - \ln x - x$ **as required** [1]
(5 marks)

(b) Find, in the form $y = mx + c$, the equation of the tangent to the curve at the point where $x = 3$.

Answer

To find the y-coordinate, substitute $x = 3$ into the equation for y found in part (a).

$$y = 5 - \ln 3 - 3 = 2 - \ln 3$$

To find the gradient at this point, substitute $x = 3$ into the equation for the gradient.

$$\frac{dy}{dx} = -1 - \frac{1}{3} = -\frac{4}{3}$$

[1]

We now have a point and the gradient of the tangent line so can find its equation using the point-gradient form, $y - y_1 = m(x - x_1)$.

$$y - (2 - \ln 3) = -\frac{4}{3}(x - 3)$$

[1]

Expand the brackets.

$$y - 2 + \ln 3 = -\frac{4}{3}x + 4$$

Collect like terms and write in the required form.

$$\mathbf{y = -\frac{4}{3}x + 6 - \ln 3}$$ [1]

(3 marks)