IGCSE · Cambridge (CIE) · Further Maths

🕐 2 hours   ❓ 27 questions

Exam Questions

Trigonometry

Trigonometric Functions / The Unit Circle / Graphs of Trigonometric Functions / Trigonometric Identities / Solving Trigonometric Equations / Trigonometric Proof

Medium (9 questions)	/37
Hard (9 questions)	/44
Very Hard (9 questions)	/62
Total Marks	**/143**

Medium Questions

1 (a) In this question, all angles are in radians.

Write down the amplitude of $2 \cos \dfrac{x}{3} - 1$.

Answer

For any function in the form $a\cos(bx) + c$, the amplitude is given by a

<div align="right">

2 [1]

(1 mark)

</div>

(b) Write down the period of $2 \cos \dfrac{x}{3} - 1$.

Answer

For any function in the form $a\cos(bx) + c$, the period is $\dfrac{2\pi}{b}$ in radians or $\dfrac{360°}{b}$ in degrees. As the units (degrees or radians) is not specified in the question you can use either
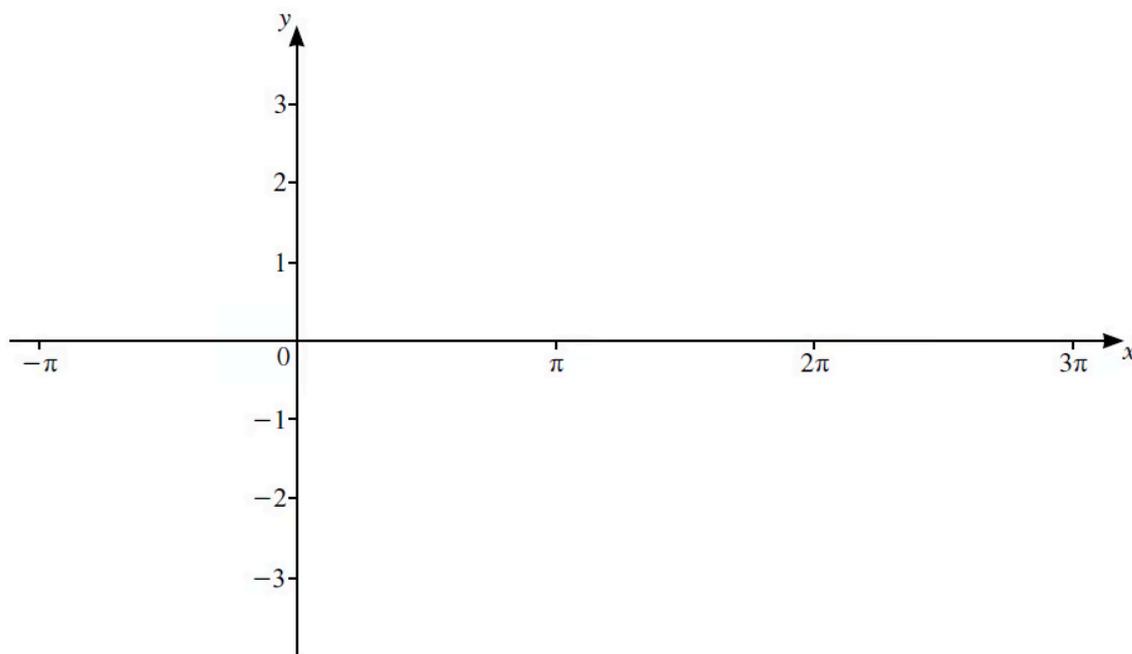
Radians:

$$\text{period} = \frac{2\pi}{\frac{1}{3}}$$
$$= 2\pi \times 3$$

Degrees:

$$\text{period} = \frac{360°}{\frac{1}{3}}$$
$$= 360° \times 3$$

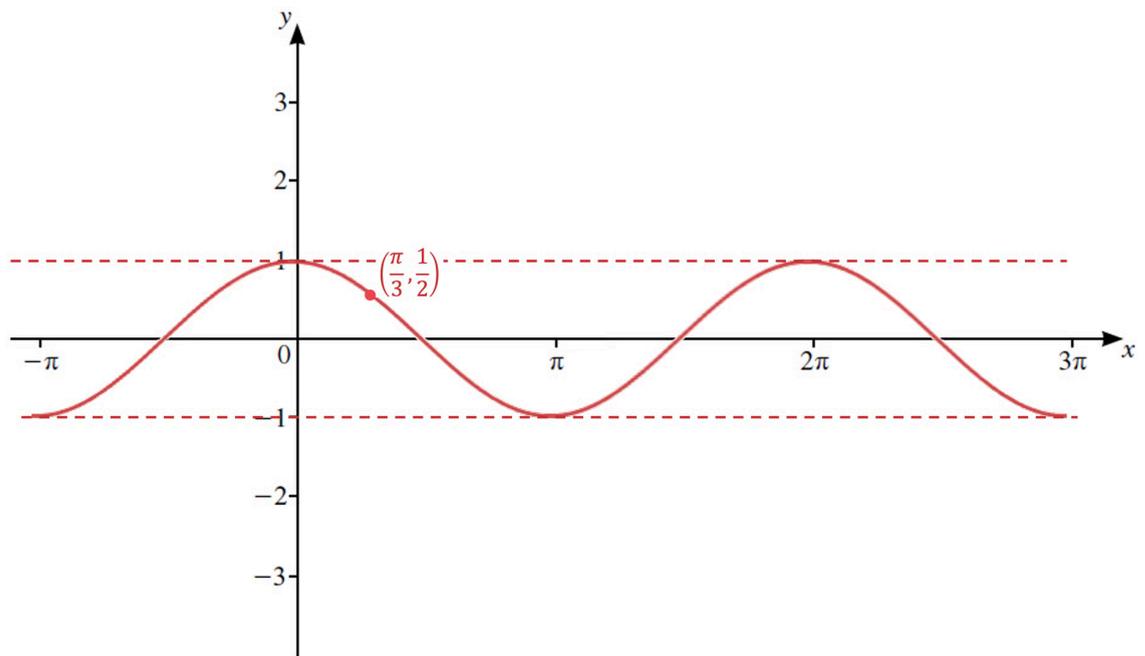<div align="right">

6π or 1080° [1]

(2 marks)

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(c) On the axes below, sketch the graph of $y = 2 \cos \dfrac{x}{3} - 1$ for $-\pi \leqslant x \leqslant 3\pi$.
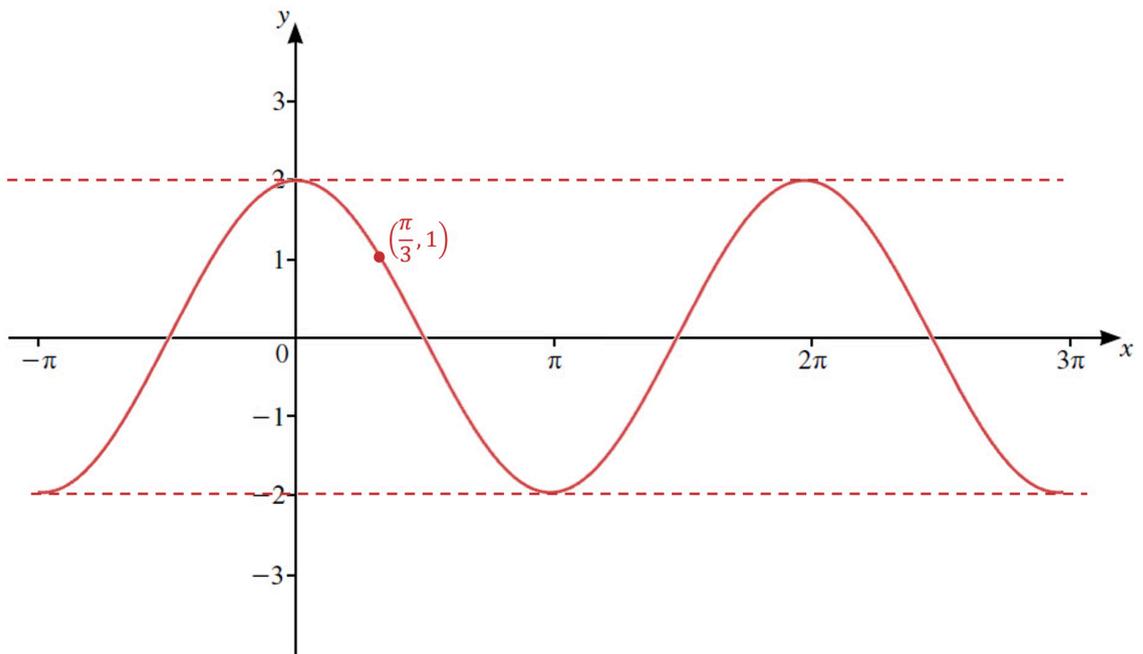

Answer

Start by sketching the graph of $y = \cos x$. Note that when $x = \dfrac{\pi}{3}$, $y = \dfrac{1}{2}$
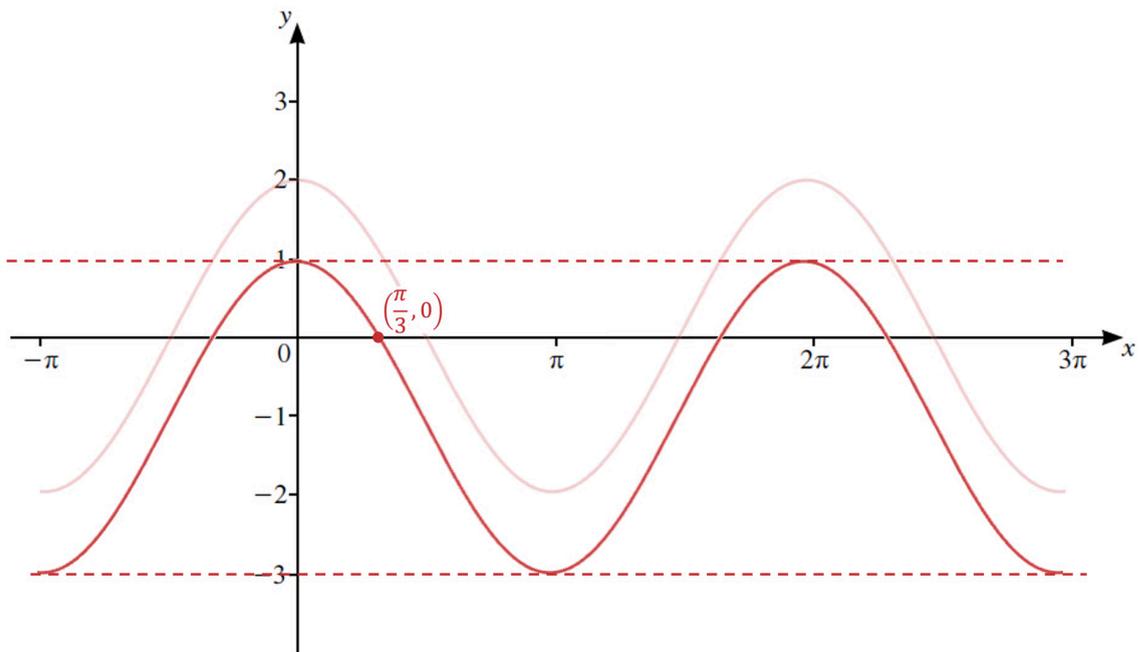

We know from part (a) that the amplitude is 2, so stretch the graph in the y-axis by 2.

Note that when $x = \dfrac{\pi}{3}$, $y = 1$

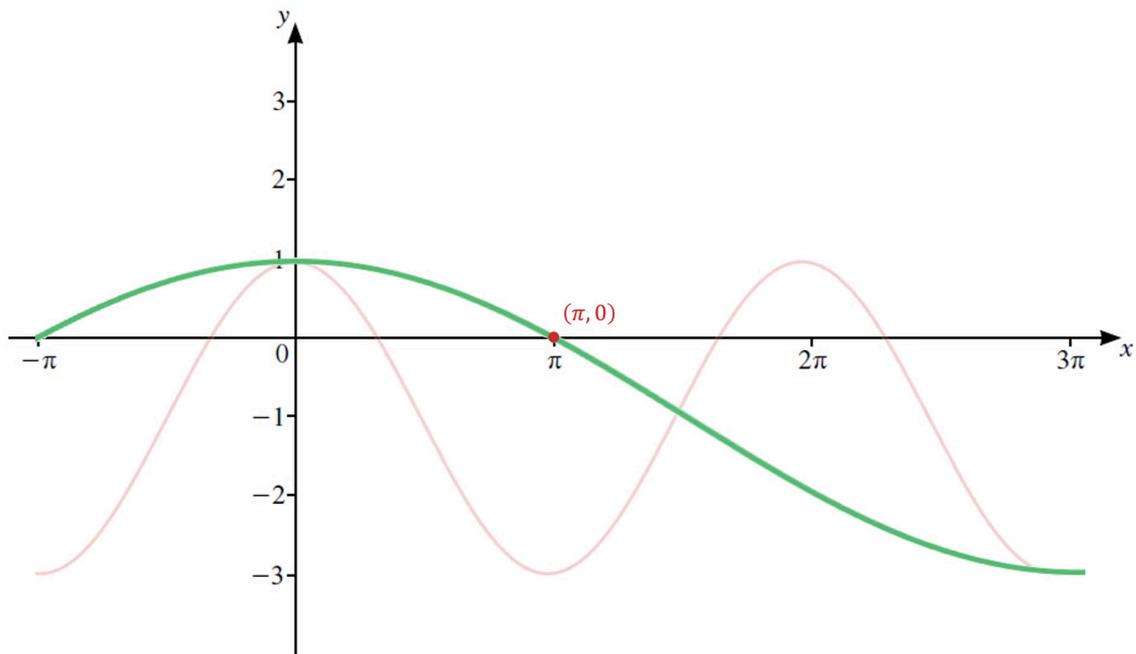The "–1" in $2\cos\dfrac{x}{3} - 1$ (or the c in $a\cos(bx) + c$) tells us that the graph of $y = 2\cos x$ is

translated down 1. Note that now, when $x = \dfrac{\pi}{3}$, $y = 0$

We know from part (b) that the period of the graph is 6π rather than 2π. Therefore the graph is stretched by a factor of 3 in the x–axis. In other words, the y–coordinates stay the same but the x–coordinates are multiplied by 3

So $\left(\dfrac{\pi}{3}, 0\right)$ becomes $(\pi, 0)$, and $(\pi, -3)$ becomes $(3\pi, -3)$

Note that due to the symmetry of the curve, if we know the curve passes through $(\pi, 0)$ then we should also have it passing through $(-\pi, 0)$

curve passing through $(-\pi, 0)$ and $(3\pi, -3)$ [1]
correct shape with maximum point on y-axis and a minimum at $x = 3\pi$ [1]
passes through $(0, 1)$ and $(\pi, 0)$ with no other positive x-intercepts [1]
(3 marks)

2 (i) Write $6xy + 3y + 4x + 2$ in the form $(ax + b)(cy + d)$, where a, b, c and d are positive integers.

(ii) Hence solve the equation $6 \sin \theta \cos \theta + 3 \cos \theta + 4 \sin \theta + 2 = 0$ for $0° < \theta < 360°$.

Answer

i) Split the expression and factorise each part.

$$6xy + 3y = 3y(2x + 1)$$

$$4x + 2 = 2(2x + 1)$$

Combine.

$$3y(2x+1)+2(2x+1)$$

Simplify.

$$(3y+2)(2x+1) \text{ [1]}$$

ii) Substitute $x = \sin(\theta)$ and $y = \cos(\theta)$ into the answer from part (i).

$$(3\cos(\theta)+2)(2\sin(\theta)+1)=0$$

Solve $3\cos(\theta)+2=0$.

$$3\cos(\theta)=-2$$

$$\theta=\cos^{-1}\left(-\frac{2}{3}\right)$$

$$\theta=131.81°$$

[1]

By using the CAST method, or by examining the Cosine curve, work out other solutions in the range $0° < \theta < 360°$

$$360-131.81$$

$$\theta=228.19°$$

[1]

Solve $2\sin(\theta)+1=0$.

$$2\sin(\theta)=-1$$

$$\theta=\sin^{-1}\left(-\frac{1}{2}\right)$$

$$\theta=-30°$$

By using the CAST method, or by examining the Sine curve, work out other solutions in the range $0° < \theta < 360°$

$$180 + 30$$

$$\theta = 210°$$

$$360 - 30$$

$$\theta = 330°$$

Give all solutions in the range.

$$\theta = 131.8°, \quad \theta = 228.2°, \quad \theta = 210°, \quad \theta = 330° \text{ [1]}$$

(5 marks)

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3 (a) The graph of $y = a + 2\tan bx$, where a and b are constants, passes through the point $(0, -4)$ and has period $480°$.

Find the value of a and of b.

Answer

Since the graph passes through $(0, -4)$, we know the the graph of $\tan x$ has been translated 4 units down.

Therefore,

$$a = -4 \text{ [1]}$$

The period of $y = \tan x$ is $180°$. Since the graph of $y = a + 2\tan bx$ has a period of $480°$,
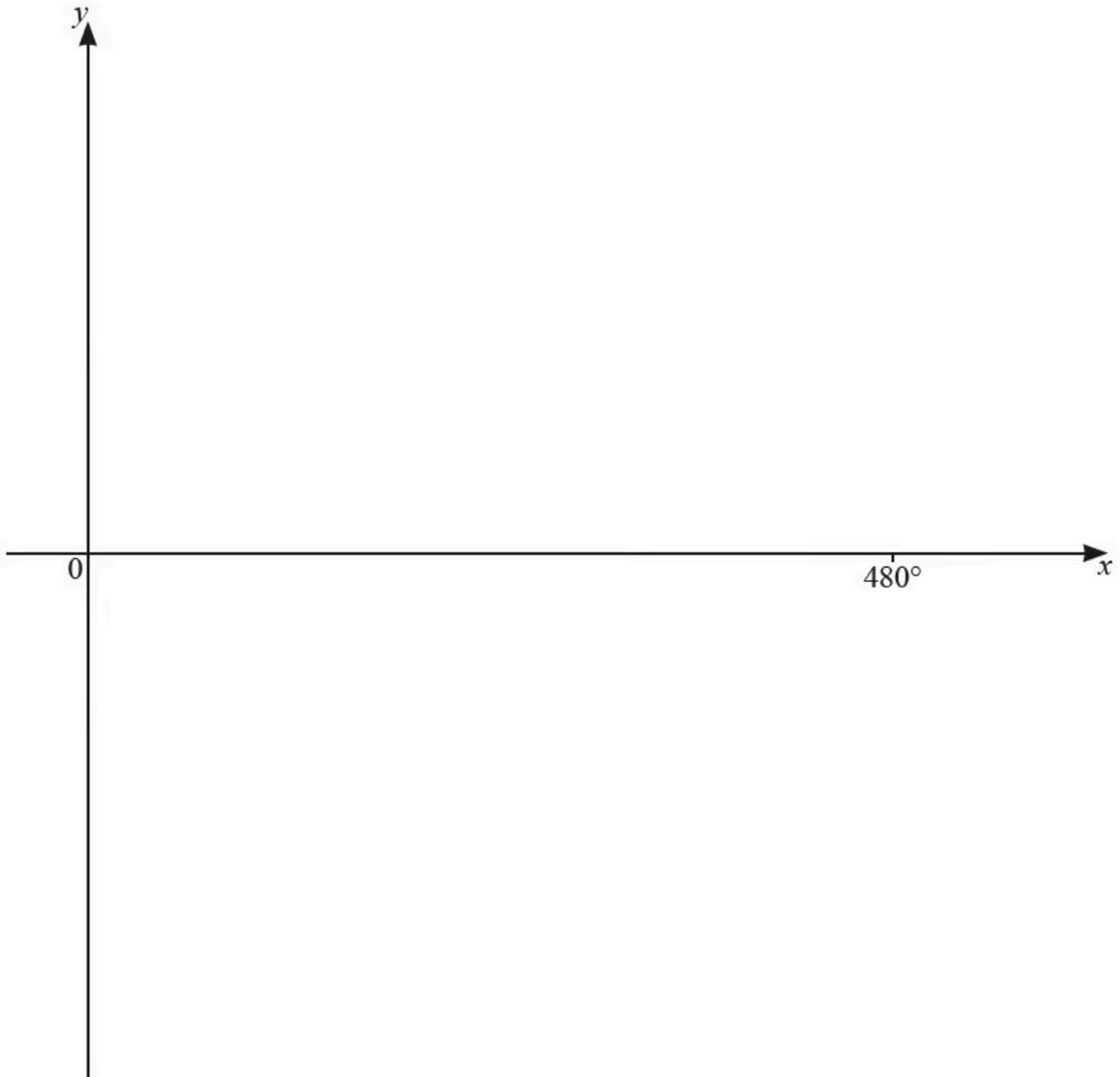
$$\frac{180}{b} = 480$$

[1]

Solve to find b.

$$b = \frac{180}{480}$$

$$b = \frac{3}{8} \quad [1]$$

(3 marks)

(b) On the axes, sketch the graph of y for values of x between 0° and 480°.

Answer

The graph passes through the point (0, –4) so use this as a starting point. The graph will be in the shape of a tan graph with a period of 480°, meaning it will have an

asymptote at $\dfrac{480}{2} = 240°$.
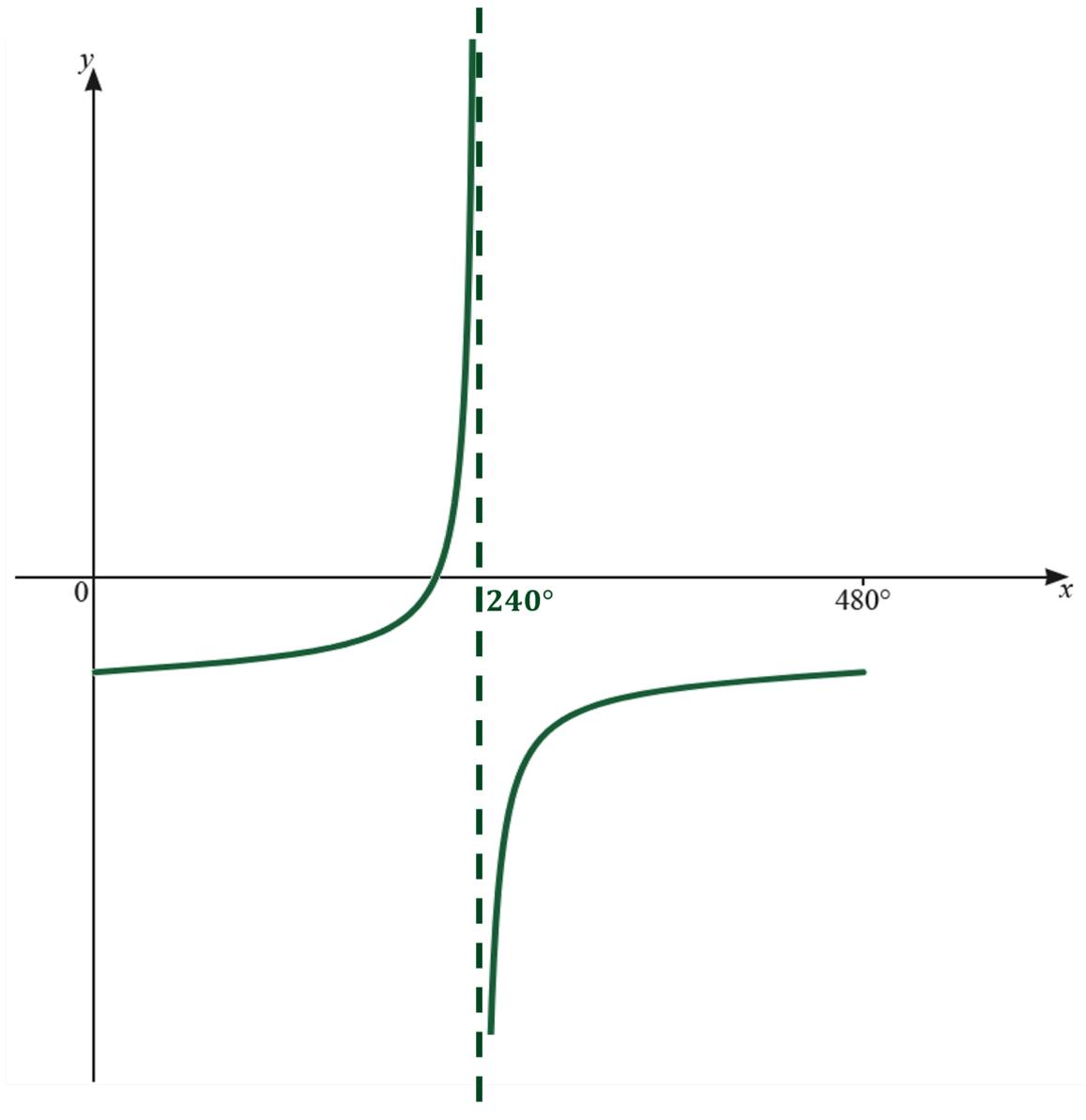
correct shape [1]

correct asymptote [1]
(2 marks)

4 Solve $\tan(\alpha + 45°) = -\dfrac{1}{\sqrt{2}}$ for $0° \leq \alpha \leq 360°$.

Answer

Before starting, note that tan is negative. We will need to look for other solutions within the range, for when tan is negative, so it is worth making a mental note of this before we begin!

Use the inverse button on your calculator to work out the principle angle

$$\alpha + 45° = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$\alpha + 45° = -35.264...$$

[1]

Use a CAST diagram to mark where tan is negative (in the sin and cos regions) and mark on the principle angle 35.264...

Look at the range and identify how many solutions you will need, bearing in mind that you still need to subtract 45 off and stay within the range $0° \leq \alpha \leq 360°$

$$\alpha + 45° = (180 - 35.264...)° \; or \; (360 - 35.264...)°$$

$$\alpha + 45° = 144.736...° \; or \; 324.736...°$$

[1]

Now subtract 45 and check that the solutions are within the range $0° \leq \alpha \leq 360°$

$$\alpha = 99.7° \; , \; 279.7°$$ [1]

(3 marks)

5
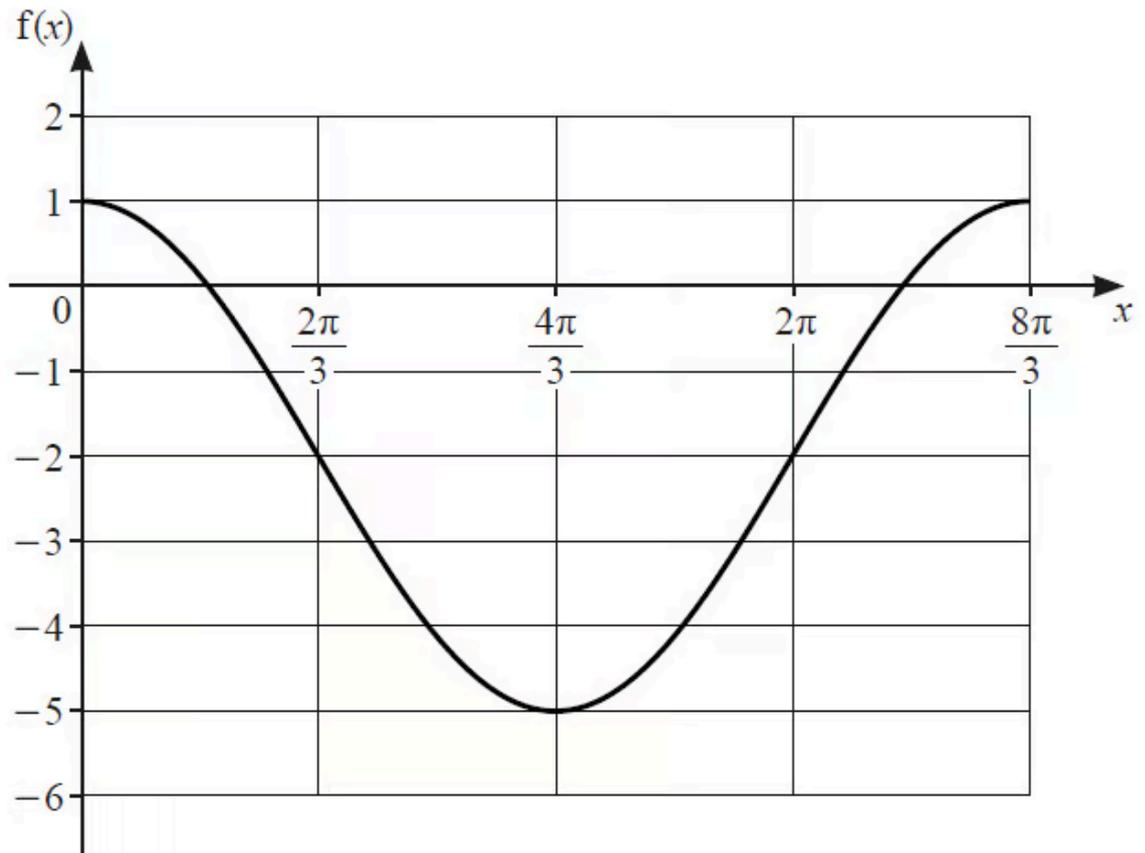
The diagram shows the graph of $f(x) = a \cos bx + c$ for $0 \leq x \leq \dfrac{8\pi}{3}$ radians.

Find the value of each of the constants a, b and c.

Answer

The coefficient of cos will apply a stretch to $y = \cos(x)$, scale factor a, parallel to the y-axis

Compare the height of the cos graph to the graph in the question and notice that it is 3 times larger

$$a = 3$$

[1]

The coefficient of x will apply a stretch to $y = \cos(x)$, scale factor $\dfrac{1}{b}$, parallel to the x-axis

To find b, compare the period of 1 wave of the given graph, to the period of $y = \cos(x)$

$$\frac{2\pi}{b} = \frac{8\pi}{3}$$

[1]

$$b = 0.75$$

[1]

This should mean that the point (0, 1) is now at (0, 3) but it is not so the graph has been translated 2 units down

The constant will translate the graph of $y = \cos(x)$ through the vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$

$$c = -2$$

$a = 3$, $b = 0.75$, $c = -2$ [1]

(4 marks)

6 The curve $y = a \sin bx + c$ has a period of 180°, an amplitude of 20 and passes through the point $(90°, -3)$. Find the value of each of the constants a, b and c.

Answer

The amplitude of the Sine graph is $|a|$

Therefore:

$a = 20$ [1]

The period of the Sine graph is $\dfrac{360°}{b}$ because bx is a horizontal stretch of scale factor

$\dfrac{1}{b}$

Therefore:

$$\frac{360°}{b} = 180°$$

$b = 2$ [1]

We now know that $a = 20$ and $b = 20$. The graph of $y = 20\sin(2x)$ will cross the x axis at $(90°, 0)$.

Since we are told the graph passes through $(90°, -3)$, we know that the graph has been translated vertically downwards by three units.

Therefore:

$c = -3$ [1]

(3 marks)

7 The function g is defined, for $-135° \leq x \leq 135°$, by $g(x) = 3\tan\dfrac{x}{2} - 4$. Sketch the graph of $y = g(x)$ on the axes below, stating the coordinates of the point where the graph crosses the y-axis.
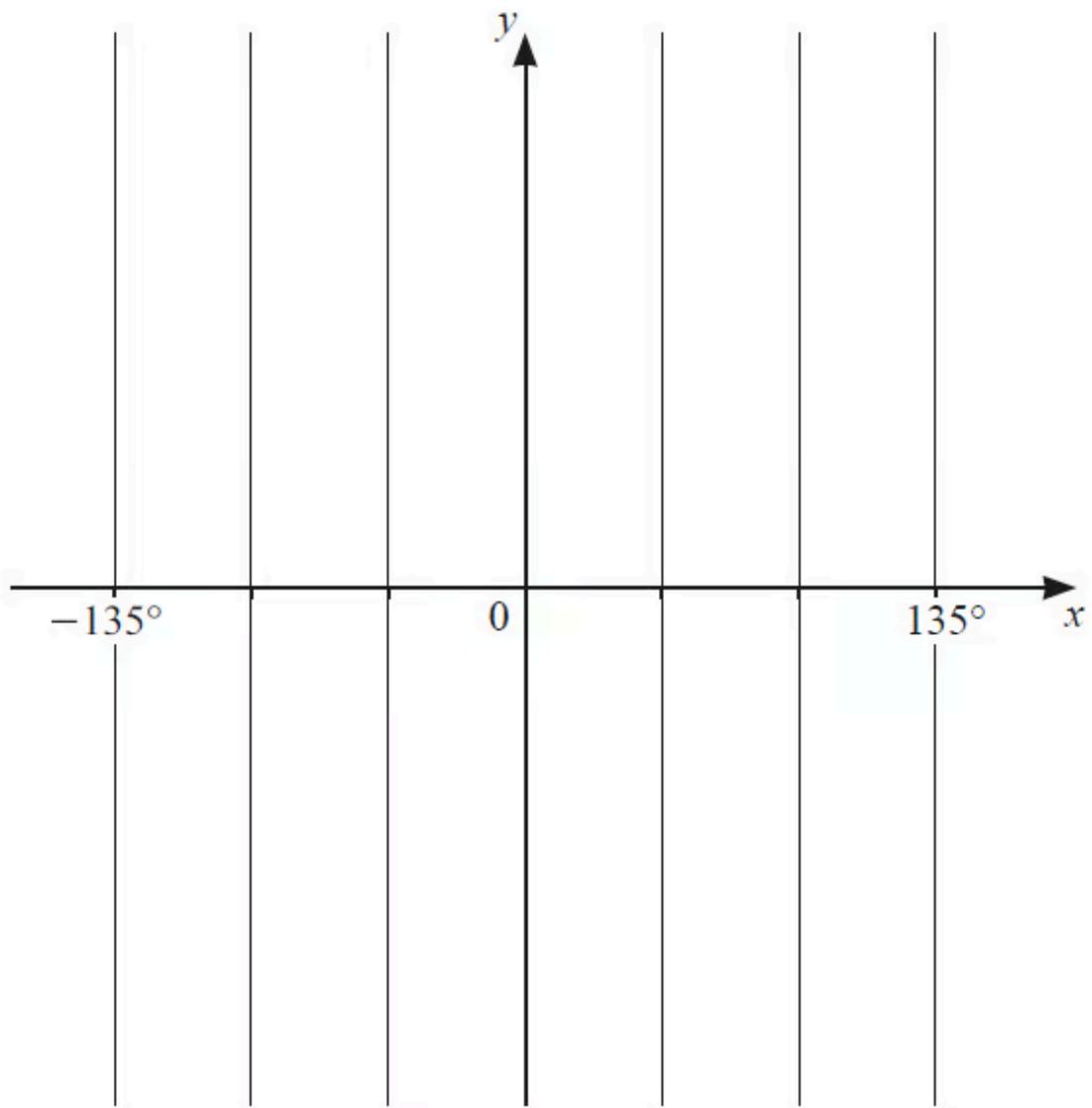
Answer

$\dfrac{x}{2}$ is a horizontal stretch of scale factor 2, therefore the asymptotes of $\pm 90°$ are now at $\pm 180°$ which is out of the given range. Because of this, we will only be drawing one continuous section.

To get the y intercept, substitute $x = 0$ into the equation.

$$y = 3\tan\left(\frac{0}{2}\right) - 4$$

$$y = -4$$

(0, -4)
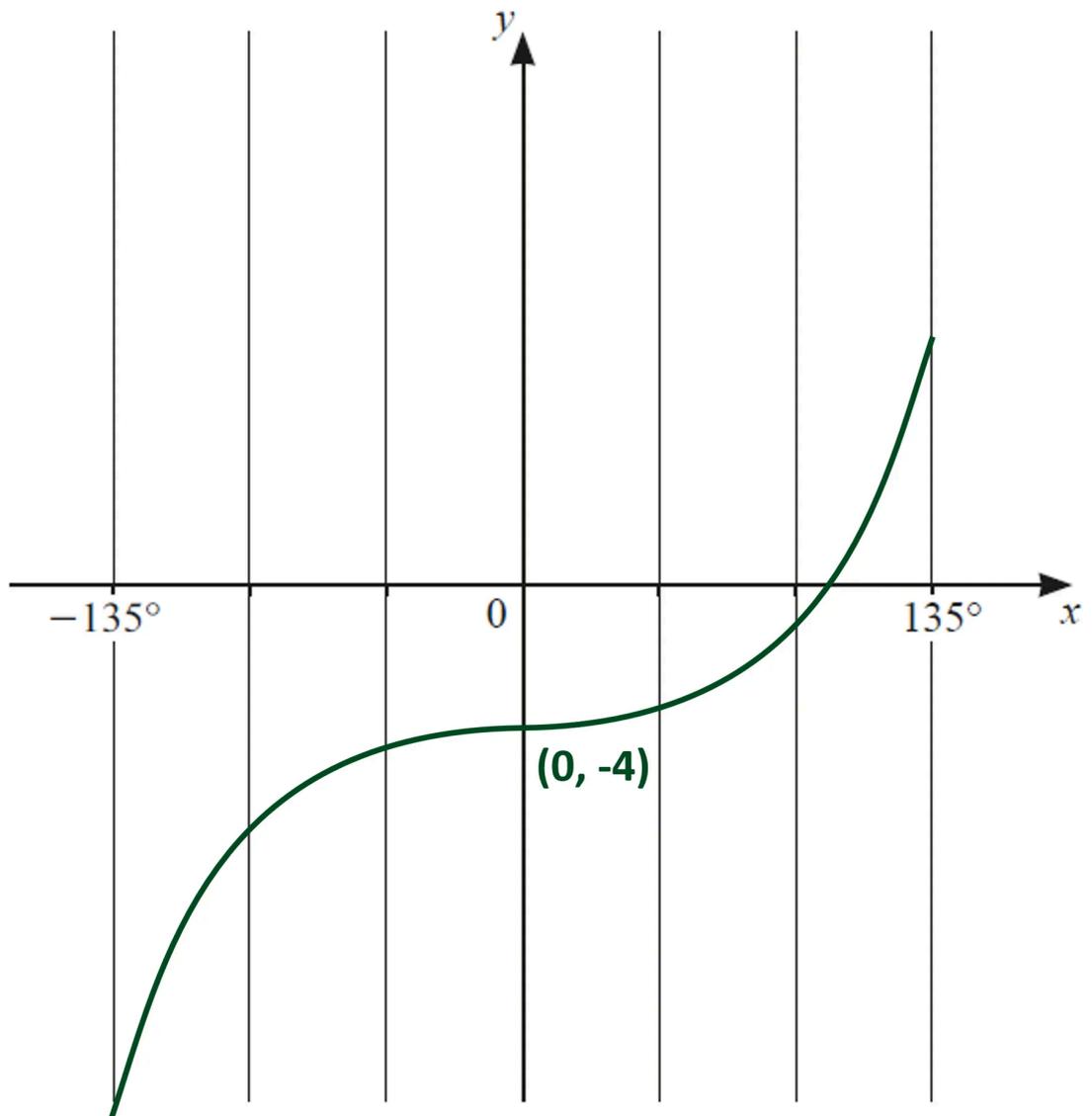
for correct tan *shape* [1]

for correct y *intercept* $(0, -4)$ [1]

(2 marks)

8 Solve the equation $5\sin\left(4B - \dfrac{\pi}{8}\right) + 2 = 0$ for $-\dfrac{\pi}{4} \le B \le \dfrac{\pi}{4}$ radians

Answer

Adjust the range $-\dfrac{\pi}{4} \le B \le \dfrac{\pi}{4}$ by multiplying the end points by 4 and subtracting $\dfrac{\pi}{8}$.

This will help us decide if our answers will be in the range needed.

$$-3.53429 \le 4B - \frac{\pi}{8} \le 2.74889$$

Rearrange the equation in the question.

$$\sin\left(4B - \frac{\pi}{8}\right) = -\frac{2}{5}$$

Inverse sine both sides.

$$4B - \frac{\pi}{8} = \sin^{-1}\left(-\frac{2}{5}\right)$$

[1]

Work out $\sin^{-1}\left(-\dfrac{2}{5}\right)$

$$\sin^{-1}\left(-\frac{2}{5}\right) = -0.4115168$$

[1]

This is within our adjusted range and so we can continue with this value.

Therefore:

$$4B - \frac{\pi}{8} = -0.4115168$$

Rearrange and solve.

$$B = -0.0047044$$

To find the other solution in the range, examine the Sine curve or use the CAST method.

$$-\pi + 0.4115168 = -2.73008$$

This is within our adjusted range and so we can continue with this value.

$$4B - \frac{\pi}{8} = -2.73008$$

Rearrange and solve.

$$B = -0.584344$$

$$B = -0.00470 \text{ [1]}$$

$$B = -0.584 \text{ [1]}$$

(4 marks)

9 (a) Write down the amplitude of $1 + 4\cos\left(\dfrac{x}{3}\right)$

Answer

The amplitude of a trigonometric function is half the distance from the highest point of the curve to the bottom point of the curve.

The graph of $y = \cos(x)$ has an amplitude of 1. The function has been translated in the y-direction, stretched in the x-direction, and stretched in the y-direction.

The stretch in the x-direction and the translation will not affect the amplitude. The stretch in the y-direction by a scale factor of 4 means that there is now a distance of 8 units between the highest and the lowest point of the curve. This means we can halve this distance to work out the amplitude.

4 [1]

(1 mark)

(b) Write down the period of $1 + 4\cos\left(\dfrac{x}{3}\right)$

Answer

The period of $y = \cos(x)$ is 360 (how often it repeats itself).

The period of the graph is only affected by any transformations in the x direction, transformations in the y direction can be ignored, as can any translations.

The graph of $y = \cos(bx)$ has a period of $\dfrac{360}{b}$, and the graph of $y = \cos\left(\dfrac{x}{3}\right)$ is a stretch in the x-direction by scale factor 3, therefore

$$360 \div \frac{1}{3} = 1080\,°$$

1080° [1]

(1 mark)

(c) On the axes below, sketch the graph of $y = 1 + 4\cos\left(\dfrac{x}{3}\right)$ for $-180° \le x° \le 180°$.
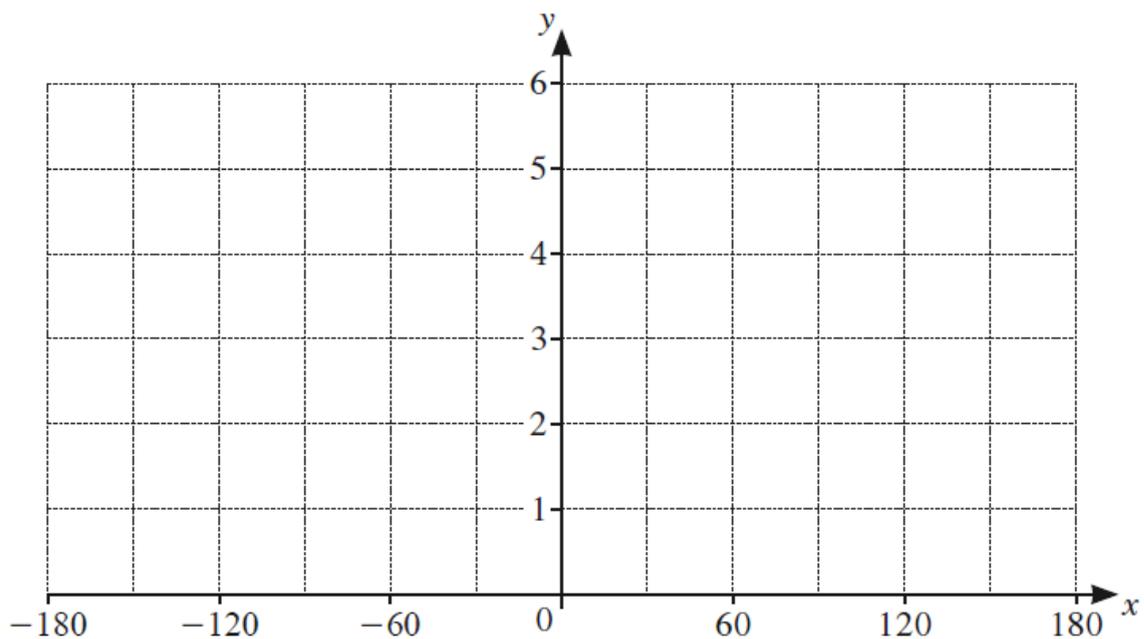
Answer

To sketch the graph, we need to apply a vertical stretch of scale factor 4, a horizontal

stretch by scale factor 3, then a vertical translation in the positive *y* direction by 1 unit
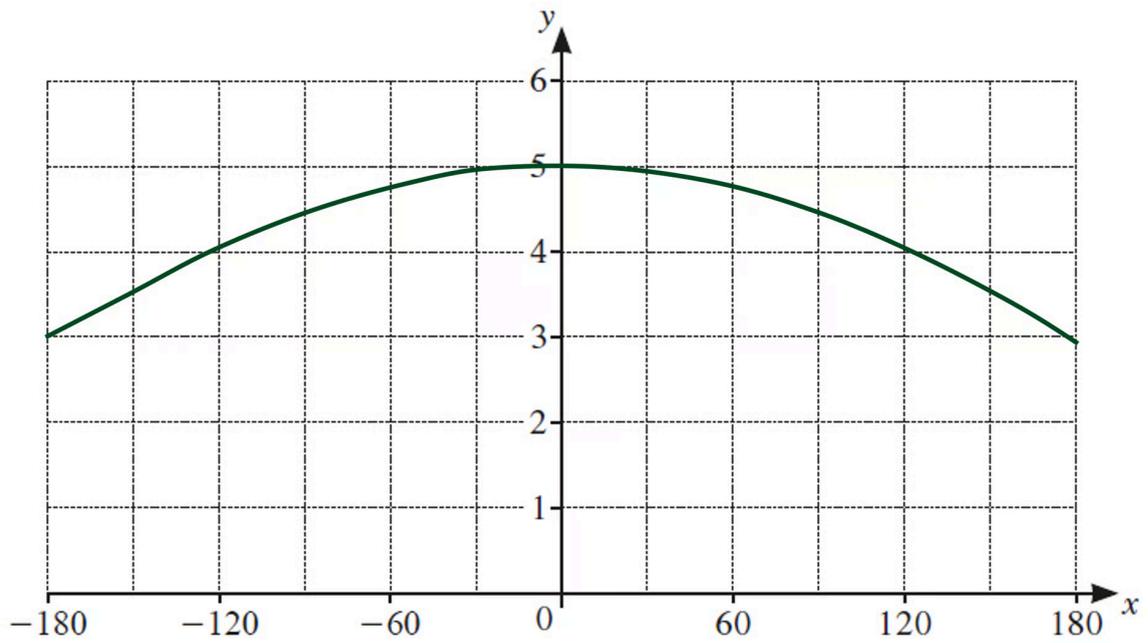
Substituting x = ±180 into the equation gives y = 3

$$1 + 4\cos\left(\frac{\pm180}{3}\right) = 3$$

symmetrical shape [1]

correct y-intercept [1]

graph through $(\pm180°, 3)$ [1]

(3 marks)

Hard Questions

1 Solve the equation $\cot\left(y - \dfrac{\pi}{2}\right) = \sqrt{3}$, where y is in radians and $0 \leqslant y \leqslant \pi$.

Answer

Rewrite the equation as an equation in tan using $\cot x = \dfrac{1}{\tan x}$

$$\frac{1}{\tan\left(y - \dfrac{\pi}{2}\right)} = \sqrt{3}$$

$$\therefore \tan\left(y - \frac{\pi}{2}\right) = \frac{1}{\sqrt{3}}$$

[1]

Use the inverse tan button on your calculator to find $y - \dfrac{\pi}{2}$ – make sure your calculator is in radians!

$$y - \frac{\pi}{2} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

[1]

Check for other solutions. Adjust the domain

$$0 \leq y \leq \pi$$

$$-\frac{\pi}{2} \leq y - \frac{\pi}{2} \leq \frac{\pi}{2}$$

$$\frac{\pi}{6} + \pi > \frac{\pi}{2} \text{ and } \frac{\pi}{6} - \pi < -\frac{\pi}{2} \therefore \text{ no other solutions within the domain}$$

So now we just need to solve $y - \dfrac{\pi}{2} = \dfrac{\pi}{6}$

$$y = \dfrac{\pi}{6} + \dfrac{\pi}{2}$$

$$\dfrac{2\pi}{3} \quad [1]$$

"2.09" or "2.09439510..." rounded correctly to at least 3 significant figures is accepted

(3 marks)

2 Solve the equation $\dfrac{1}{2}\sec\left(2\phi + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{3}}$ for $-\pi < \phi < \pi$, where ϕ is in radians. Give your answers in terms of π.

Answer

Substitute $\dfrac{1}{\cos\left(2\phi + \dfrac{\pi}{4}\right)}$ for $\sec\left(2\phi + \dfrac{\pi}{4}\right)$

$$\dfrac{1}{2} \times \dfrac{1}{\cos\left(2\phi + \dfrac{\pi}{4}\right)} = \dfrac{1}{\sqrt{3}}$$

Multiply the fractions on the left hand side.

$$\dfrac{1}{2\cos\left(2\phi + \dfrac{\pi}{4}\right)} = \dfrac{1}{\sqrt{3}}$$

Multiply both sides by 2.

$$\dfrac{1}{\cos\left(2\phi + \dfrac{\pi}{4}\right)} = \dfrac{2}{\sqrt{3}}$$

Take the reciprocal of both sides.

$$\cos\left(2\phi + \dfrac{\pi}{4}\right) = \dfrac{\sqrt{3}}{2}$$

Take $\cos^{-1}$ of both sides.

$$2\phi + \frac{\pi}{4} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$2\phi + \frac{\pi}{4} = \frac{\pi}{6}$$

Rearrange and solve to find one solution.

$$2\phi = \frac{\pi}{6} - \frac{\pi}{4}$$

$$2\phi = -\frac{\pi}{12}$$

$$\phi = -\frac{\pi}{24}$$

[1]

The range given in the question is $-\pi < \phi < \pi$. If we rewrite this for $2\phi + \frac{\pi}{4}$, our range will be

$$-\frac{7\pi}{4} < 2\phi + \frac{\pi}{4} < \frac{9\phi}{4}$$

By either using CAST method, or by examining the Cosine curve, find the other solutions to $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, ignoring the ones outside of our range.

$$\frac{13\pi}{6}, \qquad \frac{11\pi}{6}, \qquad -\frac{\pi}{6}$$

Substitute each of these values into $2\phi + \frac{\pi}{4} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, rearrange and solve.

$$2\phi + \frac{\pi}{4} = \frac{13\pi}{6}$$

$$2\phi = \frac{13\pi}{6} - \frac{\pi}{4}$$

$$\phi = \frac{23\pi}{24}$$

[1]

$$2\phi = \frac{11\pi}{6} - \frac{\pi}{4}$$

$$\phi = \frac{19\pi}{24}$$

[1]

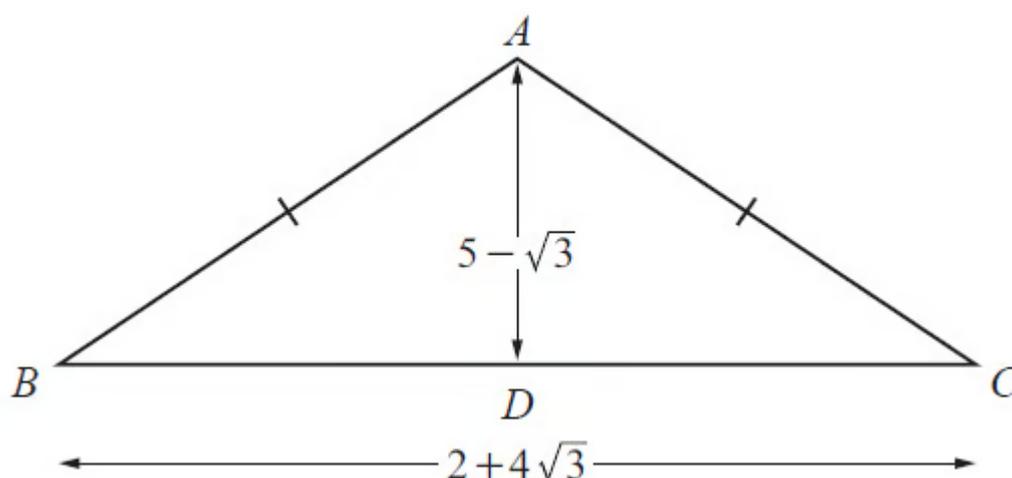$$2\phi + \frac{\pi}{4} = -\frac{\pi}{6}$$

$$2\phi = -\frac{\pi}{6} - \frac{\pi}{4}$$

$$\phi = -\frac{5\pi}{24}$$

$$\phi = -\frac{5\pi}{24}, \quad \phi = -\frac{\pi}{24}, \quad \phi = \frac{19\pi}{24}, \quad \phi = \frac{23\pi}{24}$$ [1]

(5 marks)

3 (a) In this question all lengths are in centimetres.

The diagram shows the isosceles triangle ABC, where $AB = AC$ and $BC = 2 + 4\sqrt{3}$.
The height, AD, of the triangle is $5 - \sqrt{3}$.

Find $\tan ABC$, giving your answer in the form $c + d\sqrt{3}$, where c and d are integers.

Answer

The tan ratio of a right angled triangle is given by $\tan = \dfrac{\text{opposite}}{\text{adjacent}}$

$$\text{opposite} = 5 - \sqrt{3}$$

$$\text{adjacent} = \frac{2 + 4\sqrt{3}}{2} = 1 + 2\sqrt{3}$$

$$\tan ABC = \frac{5 - \sqrt{3}}{1 + 2\sqrt{3}}$$

Now rationalise the denominator by multiplying the numerator and denominator by $1 - 2\sqrt{3}$ because this will eliminate the surd from the denominator

$$\tan ABC = \frac{5 - \sqrt{3}}{1 + 2\sqrt{3}} \times \frac{1 - 2\sqrt{3}}{1 - 2\sqrt{3}}$$

$$\tan ABC = \frac{\left(5-\sqrt{3}\right)\left(1-2\sqrt{3}\right)}{\left(1+2\sqrt{3}\right)\left(1-2\sqrt{3}\right)}$$

Expand the double brackets on the numerator and denominator

$$\tan ABC = \frac{5-10\sqrt{3}-\sqrt{3}+(2\times3)}{1-2\sqrt{3}+2\sqrt{3}-(4\times3)}$$

$$\tan ABC = \frac{5-11\sqrt{3}+6}{1-12}$$

$$\tan ABC = \frac{11-11\sqrt{3}}{-11}$$

$$\tan ABC = \sqrt{3}-1$$

$$\sqrt{3}-1 \text{[2]}$$

(2 marks)

(b) Find $\sec^2 ABC$, giving your answer in the form $e + f\sqrt{3}$, where e and f are integers.

Answer

Recall that $\sec^2 x = \tan^2 x + 1$ and use your answer from part (b) to obtain the following

$$\sec^2 ABC = \tan^2 ABC + 1$$

$$\sec^2 ABC = \left(\sqrt{3}-1\right)^2 + 1$$

[1]

Expand the double bracket

$$\sec^2 ABC = \left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right) + 1$$

Multiply out the double bracket

$$\sec^2 ABC = 3 - \sqrt{3} - \sqrt{3} + 1 + 1$$

Simplify

$$\sec^2 ABC = 5 - 2\sqrt{3}$$

$5 - 2\sqrt{3}$ [1]

(2 marks)

4 Solve $\csc\left(y + \dfrac{\pi}{3}\right) = 2$ for $0 \leq y \leq 2\pi$ radians, giving your answers in terms of π.

Answer

Using the trigonometric identity $\dfrac{1}{\sin\theta} = \csc\theta$

$$\csc\left(y + \frac{\pi}{3}\right) = 2$$

Becomes

$$\sin\left(y + \frac{\pi}{3}\right) = \frac{1}{2}$$

[1]

Taking the inverse sine of both sides gives

$$y + \frac{\pi}{3} = \frac{\pi}{6}$$

Using a CAST diagram to find the rest of the solutions gives

$$y + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

[1]

Subtracting $\dfrac{\pi}{3}$ from each of these gives

$$y = -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}$$

The interval is $0 \leq y \leq 2\pi$ radians so we can eliminate the first solution

$$y = \frac{\pi}{2}, \frac{11\pi}{6}$$

(4 marks)

5 (a) Solve $\tan 3x = -1$ for $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ radians, giving your answers in terms of π.

Answer

$$\tan 3x = -1$$

Note that tan is negative

Use tan inverse to find the principle angle

$$3x = \tan^{-1}(-1) = -\frac{\pi}{4}$$

Generate all angles, that when dividing by 3 will stay in the required range by adding and subtracting π

$$\frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{5\pi}{4}$$

[1]

Divide by 3

$$x = \frac{\pi}{4}, -\frac{\pi}{12}, \frac{5\pi}{12}$$

$$\frac{\pi}{4} \ [1]$$

$$-\frac{\pi}{12} \ [1]$$

$$\frac{5\pi}{12} \;\; [1]$$

(4 marks)

(b) Use your answers to part (a) to sketch the graph of $y = 4\tan 3x + 4$ for $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ radians on the axes below. Show the coordinates of the points where the curve meets the axes.
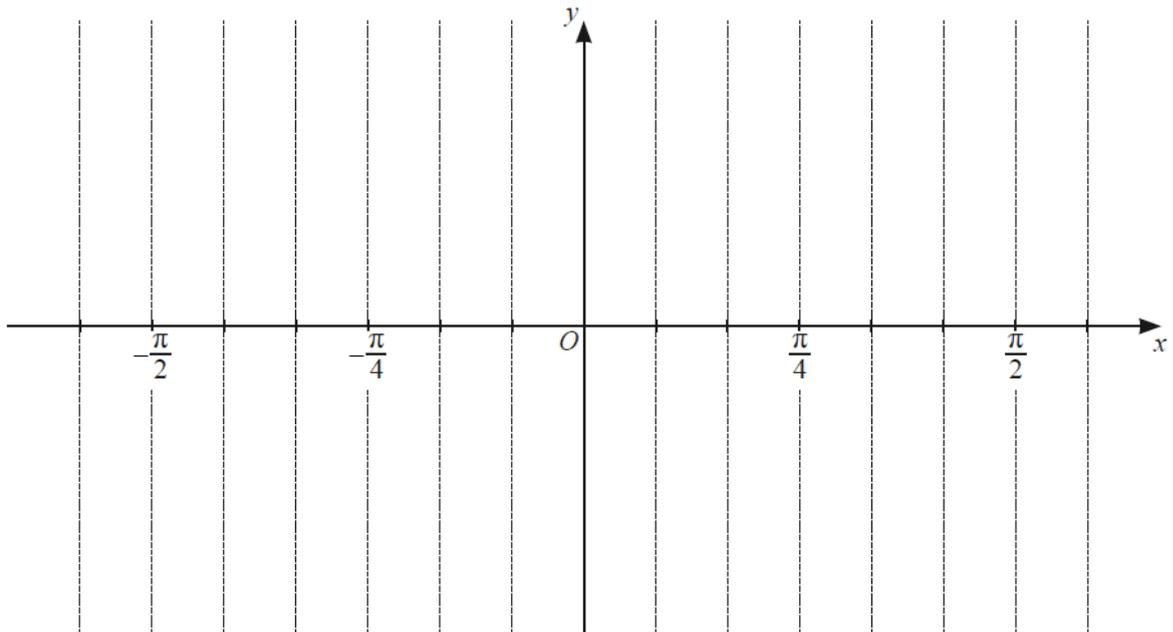
Answer

Find where the graph crosses the x-axis

Graph crosses x-axis when $y = 0$

$$0 = 4\tan 3x + 4$$

$$4\tan 3x = -4$$

$$\tan 3x = -1$$

We solved this equation in part (a) so use the solutions which are

$$x = \frac{\pi}{4}, \ -\frac{\pi}{12}, \ -\frac{5\pi}{12}$$

Find where the graph crosses the y-axis

Graph crosses y-axis when $x = 0$

$$y = 4\tan(3 \times 0) + 4 = 4$$

1. To find the asymptotes, consider how $y = \tan 3x$ differs from $y = \tan x$ (the latter has a period of π)

$y = \tan x$ has asymptotes at $\dfrac{\pi}{2}, \ -\dfrac{\pi}{2}, \ \dfrac{3\pi}{2}, \ -\dfrac{3\pi}{2}$

The 3 will stretch the graph with scale factor $\dfrac{1}{3}$ in the x direction so the period will get smaller (meaning the asymptotes are closer together)

So multiply the asymptotes of tan by $\dfrac{1}{3}$ to get the new asymptotes

$$\frac{\pi}{6},\ -\frac{\pi}{6},\frac{\pi}{2},\ -\frac{\pi}{2}$$
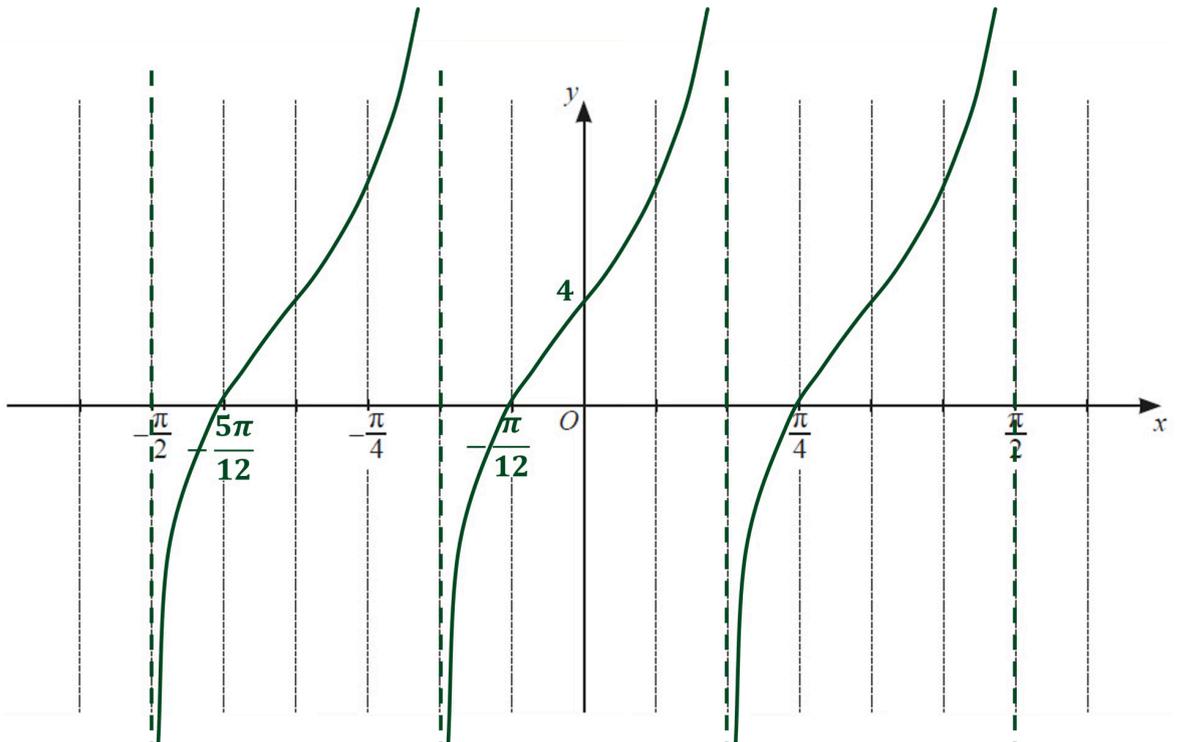
correct shape [1]

correct x-coordinates [1]

correct y-coordinates [1]

(3 marks)

6 Solve $3\cot^2 x - 14\operatorname{cosec} x - 2 = 0$ for $0° < x < 360°$.

Answer

Use the trigonometric identity $1 + \cot^2 x = \operatorname{cosec}^2 x$ to rewrite the equation

$$3(\operatorname{cosec}^2 x - 1) - 14\operatorname{cosec} x - 2 = 0$$

[1]

Expand the bracket

$$3\operatorname{cosec}^2 x - 3 - 14\operatorname{cosec} x - 2 = 0$$

$$3\text{cosec}^2x - 14\text{cosec}x - 5 = 0$$

[1]

Factorise the quadratic equation

$$(\text{cosec}x - 5)(3\text{cosec}x + 1) = 0$$

[1]

Solve the first bracket equal to 0

$$\text{either } \text{cosec}x - 5 = 0$$

$$\frac{1}{\sin x} = 5$$

$$\sin x = \frac{1}{5}$$

[1]

$$x = \sin^{-1}\left(\frac{1}{5}\right) = 11.536...$$

Find other solutions in the range

$$\text{or } x = (180 - 11.536...) = 168.463...$$

Solve the first bracket equal to 0

$$\text{or } 3\text{cosec}x + 1 = 0$$

$$\text{cosec}x = -\frac{1}{3}$$

$$\frac{1}{\sin x} = -\frac{1}{3}$$

$\sin x = -3$ which has no solutions

11.5 and 168.5 [1]

7 Solve the equation

$5 \sec^2 A + 14\tan A - 8 = 0$ for $0° \le A \le 180°$,

Answer

Rewrite $5 \sec^2 A$ using the identity $1 + \tan^2 A = \sec^2 A$

$$5(1 + \tan^2 A) + 14\tan A - 8 = 0$$

[1]

Expand and simplify.

$$5 + 5\tan^2 A + 14\tan A - 8 = 0$$
$$5\tan^2 A + 14\tan A - 3 = 0$$

Factorise.

$$(5\tan A - 1)(\tan A + 3) = 0$$

[1]

Solve.

$$\tan A = \frac{1}{5} \text{ and } \tan A = -3$$

$$A = \tan^{-1}\left(\frac{1}{5}\right) \text{ and } A = \tan^{-1}(-3)$$

$$A = 11.30993247 \text{ and } A = -71.56505118$$

One of the solutions is in the range $0° \le A \le 180°$

The other is not in the correct range, and any other solutions are out of range.

The graph of $\tan A$ has a period of $180°$ and so

$$A = -71.6 + 180$$
$$A = 108.4°$$

$$A = 11.3°\ [1]$$

$$A = 108.4°\ [1]$$

(4 marks)

8 Solve $\sin\left(3\phi + \dfrac{2\pi}{3}\right) = \cos\left(3\phi + \dfrac{2\pi}{3}\right)$ for $0 \le \phi \le \dfrac{2\pi}{3}$ radians, giving your answers in terms of π.

Answer

Using the identity that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ and dividing both sides by $\cos\theta$ we have

$$\tan\left(3\phi + \frac{2\pi}{3}\right) = 1$$

[1]

Taking the inverse of each side gives our first solution

$$3\phi + \frac{2\pi}{3} = \frac{\pi}{4}$$

We will need to expand our interval before using a CAST diagram, so we will multiply the interval by 3, then add $\dfrac{2\pi}{3}$. Using a CAST diagram with our new interval to find the solutions between $\dfrac{2\pi}{3} \le 3\phi + \dfrac{2\pi}{3} \le \dfrac{8\pi}{3}$, we get

$$3\phi + \frac{2\pi}{3} = \frac{5\pi}{4}, \frac{9\pi}{4}$$

[1]

Subtracting $\dfrac{2\pi}{3}$ and dividing by 3 gives

$$\phi = \frac{7\pi}{36} \quad [1]$$

$$\phi = \frac{19\pi}{36} \quad [1]$$

(4 marks)

9 (a) In this question, all angles are measured in radians.

The graph of $y = a\sin bx + c$ has an amplitude of 10, a period of 16π and passes through the point with coordinates $(12\pi, 4)$.

Find the constants a, b and c.

Answer

For the graph $y = a\sin bx + c$ the amplitude is a

$$a = 10$$

[B1]

For the graph $y = a\sin bx + c$ the period is $\dfrac{2\pi}{b}$ (when using radians)

$$\frac{2\pi}{b} = 16\pi$$

Solve for b

$$2 = 16b$$

$$b = \frac{1}{8}$$

[B1]

Substitute $x = 12\pi$ and $y = 4$ into $y = 10\sin\left(\dfrac{x}{8}\right) + c$

$$4 = 10\sin\left(\frac{12\pi}{8}\right) + c$$

Mark Scheme and Guidance

This mark is for substituting $x = 12\pi$ and $y = 4$ into your $y = a\sin bx + c$ (a, b and c do not have to be correct to get this mark).

Simplify and use the exact trig value that $\sin\left(\dfrac{3\pi}{2}\right) = -1$

$$4 = 10\sin\left(\dfrac{3\pi}{2}\right) + c$$
$$4 = 10(-1) + c$$
$$14 = c$$

Present all the solutions from above

$a = 10$, $b = \dfrac{1}{8}$ and $c = 14$

(4 marks)

(b) Sketch the graph of $y = \left|\tan\left(\dfrac{x}{2}\right) + 1\right|$ for $-\pi < x < \pi$.

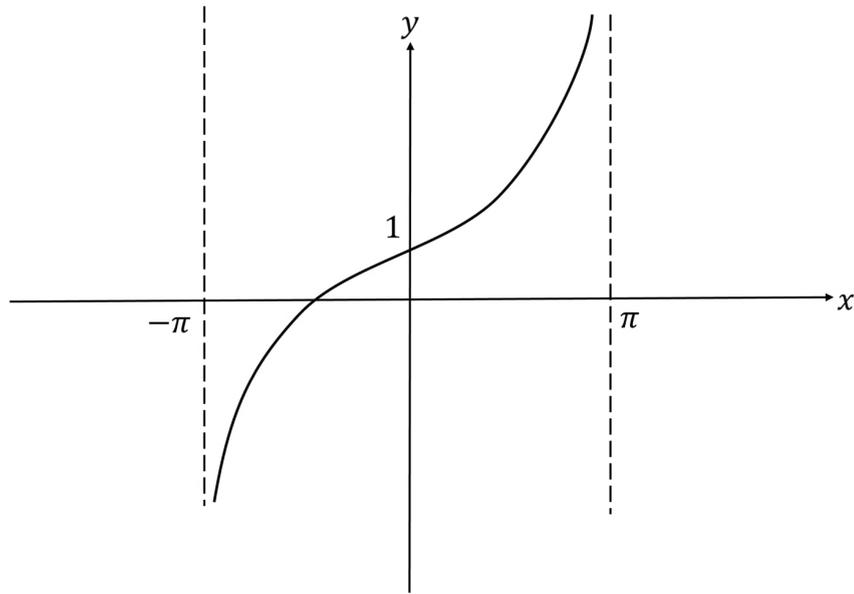Label clearly any points of intersection with the coordinate axes and any asymptotes.

Answer

The graph of $y = \tan\left(\dfrac{x}{2}\right)$ is the graph of $y = \tan x$ after a horizontal stretch of scale factor 2 (i.e. has asymptotes at $x = \pm\pi$)

The graph of $y = \tan\left(\dfrac{x}{2}\right) + 1$ is the graph of $y = \tan\left(\dfrac{x}{2}\right)$ translated upwards by 1 unit (asymptotes are not affect)

Sketch the graph of $y = \tan\left(\dfrac{x}{2}\right) + 1$

Find the coordinate of the x-intercept (by setting $y = 0$ and solving)

$$\tan\left(\frac{x}{2}\right) + 1 = 0$$

$$\tan\left(\frac{x}{2}\right) = -1$$

[M1]

Use the exact trig value $\tan\left(-\dfrac{\pi}{4}\right) = -1$ to solve for x

$$\frac{x}{2} = -\frac{\pi}{4}$$

$$x = -\frac{\pi}{2}$$

[A1]

Apply the modulus transformation to $y = \tan\left(\dfrac{x}{2}\right) + 1$ (reflecting any part below the axis)

$$y = \left|\tan\left(\dfrac{x}{2}\right) + 1\right|$$
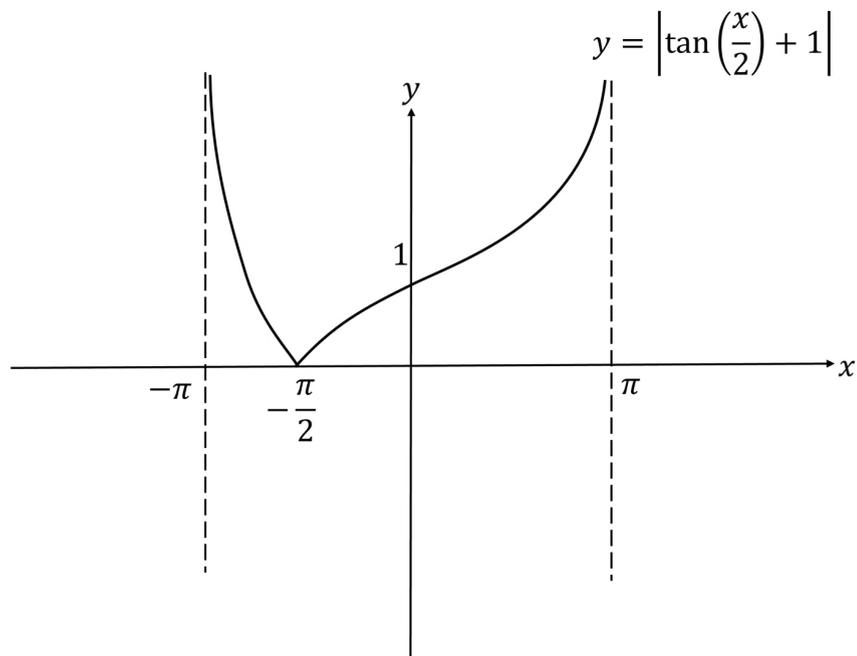
Mark Scheme and Guidance

B1: For drawing the correct shape between 2 vertical asymptotes (the numerical values may be incorrect for this mark).

B1: For drawing the shape between two vertical asymptotes with all numerical values labelled clearly and correctly.

(4 marks)

Very Hard Questions

1 (i) Show that $\dfrac{\cos^2 2x}{1+\sin 2x} = 1 - \sin 2x$.

(ii) Hence solve the equation for $\dfrac{3\cos^2 2x}{1+\sin 2x} = 1$ for $0° \leqslant x \leqslant 90°$.

Answer

i) We can rewrite the numerator using the Pythagorean trig identity $\cos^2 A = 1 - \sin^2 A$ ($\sin^2 A + \cos^2 A = 1$ is given on the formula sheet)

$$\frac{\cos^2 2x}{1+\sin 2x} = \frac{1-\sin^2 2x}{1+\sin 2x}$$

[1]

We can rewrite the numerator using the difference of two squares and cancel the fraction

$$\frac{(1+\sin 2x)(1-\sin 2x)}{1+\sin 2x} = 1-\sin 2x \text{ [1]}$$

ii)   The left hand side of the equation is $3\left(\dfrac{\cos^2 2x}{1+\sin 2x}\right)$. Therefore

$$\frac{3\cos^2 2x}{1+\sin 2x} = 3(1-\sin 2x) = 1$$

Now solve

$$1-\sin 2x = \frac{1}{3}$$

$$\sin 2x = \frac{2}{3}$$

[1]

The domain is given in degrees so make sure your calculator is in degrees

$$2x = \sin^{-1}\left(\frac{2}{3}\right) = 41.81031...°$$

Divide both sides by two to get the first solution

$$x_1 = 20.905157...$$

[1]

The domain is for x so double it to get the domain for $2x$

$$0° \leq 2x \leq 180°$$

Use the trig identity $\sin A = \sin(180° - A)$ to find the second solution

$$2x_2 = 180° - 41.81031...° = 138.189685...°$$

Divide by 2

$$x_2 = 69.09484...$$

$$20.9°, \quad 69.1° \; (3 \text{ s.f.}) \; [2]$$

one mark for each correct answer correctly rounded to at least 3 significant figures

(6 marks)

2 (i) Show that $\dfrac{1}{\sin \theta - 1} - \dfrac{1}{\sin \theta + 1} = a \sec^2 \theta$, where a is a constant to be found.

[3]

(ii) Hence solve $\dfrac{1}{\sin 3\phi - 1} - \dfrac{1}{\sin 3\phi + 1} = -8$ for $-\dfrac{\pi}{3} \leq \phi \leq \dfrac{\pi}{3}$ radians.

[5]

Answer

i) Find a common denominator which will be $(\sin\theta - 1)(\sin\theta + 1)$

To write the two fractions over this common denominator, we will need to multiply the numerator and denominator of the first fraction by $(sin\theta + 1)$ and the numerator and the denominator of the second fraction by $(\sin\theta - 1)$ giving

$$\frac{\sin\theta + 1}{(\sin\theta - 1)(\sin\theta + 1)} - \frac{\sin\theta - 1}{(\sin\theta + 1)(\sin\theta - 1)}$$

Combine the two separate fractions into one fraction now, since they have the same denominator

$$= \frac{\sin\theta + 1 - (\sin\theta - 1)}{(\sin\theta - 1)(\sin\theta + 1)}$$

[1]

Simplify the numerator by expanding the bracket

$$= \frac{\sin\theta + 1 - \sin\theta + 1}{(\sin\theta - 1)(\sin\theta + 1)}$$

Now we can cancel the $\sin\theta$ terms

$$= \frac{2}{(\sin\theta - 1)(\sin\theta + 1)}$$

Now we can notice that the denominator is the difference of two squares. Use that $(a - b)(a + b) = a^2 - b^2$

$$= \frac{2}{\sin^2\theta - 1}$$

Since $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta - 1 = -\cos^2\theta$

$$= \frac{2}{-\cos^2\theta}$$

[1]

$$= -2\sec^2\theta$$

$a = -2$ [1]

ii) Use part (a) with $\theta = 3\phi$

$$\frac{1}{\sin 3\phi - 1} - \frac{1}{\sin 3\phi + 1} = -2\sec^2 3\phi$$

The right-hand side is equal to –8 (from the question) so

$$-2\sec^2 3\phi = -8$$

Solve

$$\sec^2 3\phi = 4$$

Square root

$$\sec 3\phi = \pm 2$$

[1]

Recall that $\sec x = \dfrac{1}{\cos x}$

$$\frac{1}{\cos 3\phi} = \pm 2$$

$$\cos 3\phi = \pm \frac{1}{2}$$

[1]

Use cos inverse on your calculator in radian mode and use a CAST diagram to find all the angles required. Remember that we will still need to divide by 3 and they should remain in the given range.

$$3\phi = -\frac{2\pi}{3}, \ -\frac{\pi}{3}, \ \frac{\pi}{3}, \ \frac{2\pi}{3}$$

[1]

Divide by 3

$$\phi = -\frac{2\pi}{9}, -\frac{\pi}{9}, \frac{\pi}{9}, \frac{2\pi}{9}$$

$$-\frac{2\pi}{9}, -\frac{\pi}{9}, \frac{\pi}{9}, \frac{2\pi}{9} \quad [2]$$

(8 marks)

3 (i) Show that $\dfrac{1}{\sec\theta - 1} - \dfrac{1}{\sec\theta + 1} = 2\cot^2\theta$.

[3]

(ii) Hence solve $\dfrac{1}{\sec 2x - 1} - \dfrac{1}{\sec 2x + 1} = 6$ for $-90° < x < 90°$.

[5]

Answer

i) Looking at the left hand side and finding a common denominator gives

$$\frac{1}{\sec\theta - 1} - \frac{1}{\sec\theta + 1} = \frac{\sec\theta + 1 - (\sec\theta - 1)}{(\sec\theta - 1)(\sec\theta + 1)}$$

[1]

Expanding the brackets and simplifying

$$\frac{2}{\sec^2\theta - 1}$$

Using the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ we have

$$\frac{2}{\tan^2\theta}$$

[1]

Using the trigonometric identity $\dfrac{1}{\tan\theta} = \cot\theta$ we have an expression which is equal to the right hand side, as required

$$\frac{2}{\tan^2\theta} = 2\cot^2\theta$$

2cot$^2\theta$ [1]

ii) Comparing to part (i) but with $\theta = 2x$ we have

$$2\cot^2 2x = 6$$

[1]

$$\cot^2 2x = 3$$

Taking the square root of both sides

$$\cot 2x = \pm\sqrt{3}$$

Using the trigonometric identity $\cot x = \dfrac{1}{\tan x}$ we have

$$\frac{1}{\tan 2x} = \pm\sqrt{3}$$

$$\pm\frac{1}{\sqrt{3}} = \tan 2x$$

[1]

Using a calculator to find the solutions

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30,\ \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30$$

[1]

Using the tan graph, finding the solutions for $2x$ means using the interval $-180° < 2x < 180°$.

In general, if $x°$ is an angle that solves $\tan(x) = k$, then $x° \pm 180°$ is another angle that solves the same equation, therefore

$$2x = -150°, -30°, 30°, 150°$$

$$x = -75°, -15°, 15°, 75°$$

1 mark for each correct pair of solutions [2]

(8 marks)

4 Show that $\dfrac{\sin^4 y - \cos^4 y}{\cot y} = \tan y - 2\cos y \sin y$.

Answer

Start with the left hand side and factorise the difference of two squares

$$\frac{\sin^4 y - \cos^4 y}{\cot y} = \frac{(\sin^2 y - \cos^2 y)(\sin^2 y + \cos^2 y)}{\cot y}$$

[1]

$\cot y \equiv \dfrac{1}{\tan y}$ so

$$= (\sin^2 y - \cos^2 y)(\sin^2 y + \cos^2 y) \times \tan y$$

[1]

$\sin^2 y + \cos^2 y \equiv 1$ so

$$= (\sin^2 y - \cos^2 y) \times \tan y$$

[1]

$$= ((1 - \cos^2 y) - \cos^2 y) \times \tan y$$

$$= (1 - 2\cos^2 y)\frac{\sin y}{\cos y}$$

Expand the bracket

$$= \frac{\sin y}{\cos y} - 2\sin y \cos y$$

$$= \tan y - 2\sin y \cos y$$

$$\tan y - 2\sin y \cos y \, [1]$$

(4 marks)

5 (i) Show that $\dfrac{1}{(1 + \operatorname{cosec} \theta)(\sin \theta - \sin^2 \theta)} = \sec^2 \theta$.

[4]

(ii) Hence solve $(1 + \operatorname{cosec} \theta)(\sin \theta - \sin^2 \theta) = \dfrac{3}{4}$ for $-180° \le \theta \le 180°$.

[4]

Answer

i) Looking only at the left-hand side of the equation, and using the reciprocal trig function $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$, we have

$$\frac{1}{\left(1 + \dfrac{1}{\sin \theta}\right)(\sin \theta - \sin^2 \theta)}$$

[1]

Expanding the brackets gives

$$\frac{1}{\sin \theta - \sin^2 \theta + 1 - \sin \theta}$$

[1]

Simplifying gives

$$\frac{1}{1 - \sin^2\theta}$$

Using the identity $1 - \cos^2\theta \equiv \sin^2\theta$

$$\frac{1}{\cos^2\theta}$$

[1]

We know that $\dfrac{1}{\cos\theta} = \sec\theta$, therefore

$$\frac{1}{\cos^2\theta} = \sec^2\theta$$

$\sec^2\theta$ as required [1]

ii) Using part (i), and the reciprocal trig function $\dfrac{1}{\cos^2\theta} = \sec^2\theta$, we have

$$\cos^2\theta = \frac{3}{4}$$

[1]

Finding the square root of both sides

$$\cos\theta = \frac{\pm\sqrt{3}}{2}$$

[1]

Using a calculator to find the first solutions, we find that

$$\theta = 30°, 150°$$

[1]

Using a CAST diagram, given that the required interval is $-180° \leq \theta \leq 180°$, we can find the other solutions

$$\theta = -150°, -30°, 30°, 150°\,[1]$$

(8 marks)

6 Show that $\dfrac{\sin x \tan x}{1 - \cos x} = 1 + \sec x$.

Answer

Start with the left hand side and use $\tan x = \dfrac{\sin x}{\cos x}$

$$\frac{\sin x \tan x}{1 - \cos x} = \frac{\sin x \times \dfrac{\sin x}{\cos x}}{1 - \cos x}$$

[1]

Multiply the numerator and denominator by $\cos x$

$$= \frac{\sin^2 x}{\cos x (1 - \cos x)}$$

Use $\sin^2 x + \cos^2 x \equiv 1$ in the numerator

$$= \frac{1 - \cos^2 x}{\cos x (1 - \cos x)}$$

[1]

Factorise the numerator as it is the difference of 2 squares

$$= \frac{(1 - \cos x)(1 + \cos x)}{\cos x (1 - \cos x)}$$

[1]

$$= \frac{1 + \cos x}{\cos x}$$

Separate the fraction into two fractions

$$= \frac{1}{\cos x} + \frac{\cos x}{\cos x}$$

$$= \sec x + 1$$

$$\therefore \frac{\sin x \tan x}{1 - \cos x} = 1 + \sec x$$

[1]

(4 marks)

7 (a) Given that $2 \cos x = 3 \tan x$, show that $2 \sin^2 x + 3 \sin x - 2 = 0$.

Answer

Substitute $\tan x = \frac{\sin x}{\cos x}$ into the equation.

$$2 \cos x = 3 \left(\frac{\sin x}{\cos x} \right)$$

Multiply through by $\cos x$.

$$2 \cos^2 x = 3 \sin x$$

[1]

Substitute $\cos^2 x = 1 - \sin^2 x$ into the equation.

$$2(1 - \sin^2 x) = 3 \sin x$$

[1]

Expand the brackets and rearrange.

$$2 - 2 \sin^2 x = 3 \sin x$$
$$- 2 \sin^2 x - 3 \sin x + 2 = 0$$

Multiply through by –1.

$$2 \sin^2 x + 3 \sin x - 2 = 0 \quad [1]$$

(b) Hence solve $2\cos\left(2\alpha + \dfrac{\pi}{4}\right) = 3\tan\left(2\alpha + \dfrac{\pi}{4}\right)$ for $0 < \alpha < \pi$ radians, giving your answers in terms of π.

Answer

Factorise and solve $2\sin^2 x + 3\sin x - 2 = 0$

$$(2\sin x - 1)(\sin x + 2) = 0$$

$$\sin x = \frac{1}{2} \text{ and } \sin x = -2$$

Only one of these solutions will give us an answer, since we cannot take the inverse sine of a number smaller than –1.

x has been replaced with $2\alpha + \dfrac{\pi}{4}$

Therefore,

$$\sin\left(2\alpha + \frac{\pi}{4}\right) = \frac{1}{2}$$

[1]

The range is given as $0 < \alpha < \pi$. Edit this to make α into $2\alpha + \dfrac{\pi}{4}$ by multiplying both sides of the inequality by 2 and adding $\dfrac{\pi}{4}$.

$$\frac{\pi}{4} < 2\alpha + \frac{\pi}{4} < \frac{9}{4}\pi$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{6}\pi.$$

By using the CAST method, or by examining the sine graph, identify other solutions. Discard any which are out of the adjusted range.

$$2\alpha + \frac{\pi}{4} = \frac{5\pi}{6} \qquad \text{and} \qquad 2\alpha + \frac{\pi}{4} = \frac{13\pi}{6}$$

Subtract $\frac{\pi}{4}$ from both.

$$2\alpha = \frac{7\pi}{12} \qquad and \qquad 2\alpha = \frac{23\pi}{12}$$

[1]

Divide by 2.

$$\alpha = \frac{7\pi}{24} \quad [1]$$

$$\alpha = \frac{23\pi}{24} \quad [1]$$

(4 marks)

8 Solve the equation $5 \tan x - 3 \cot x = 2 \sec x$ for $0° \leq x \leq 360°$.

Answer

$$5\tan x - 3\cot x = 2\sec x$$

Replace tan, cot and sec

$$5\frac{\sin x}{\cos x} - 3\frac{\cos x}{\sin x} = \frac{2}{\cos x}$$

[1]

Multiply through by $\sin x \cos x$ to get rid of the denominators

$$5\sin^2 x - 3\cos^2 x = 2\sin x$$

Replace $\cos^2 x$ using $\sin^2 x + \cos^2 x \equiv 1$

$$5\sin^2 x - 3(1 - \sin^2 x) = 2\sin x$$

[1]

Multiply out the bracket

$$5\sin^2 x - 3 + 3\sin^2 x = 2\sin x$$

Subtract $2\sin x$ to make the quadratic equal to 0 and simplify

$$8\sin^2 x - 2\sin x - 3 = 0$$

[1]

Factorise

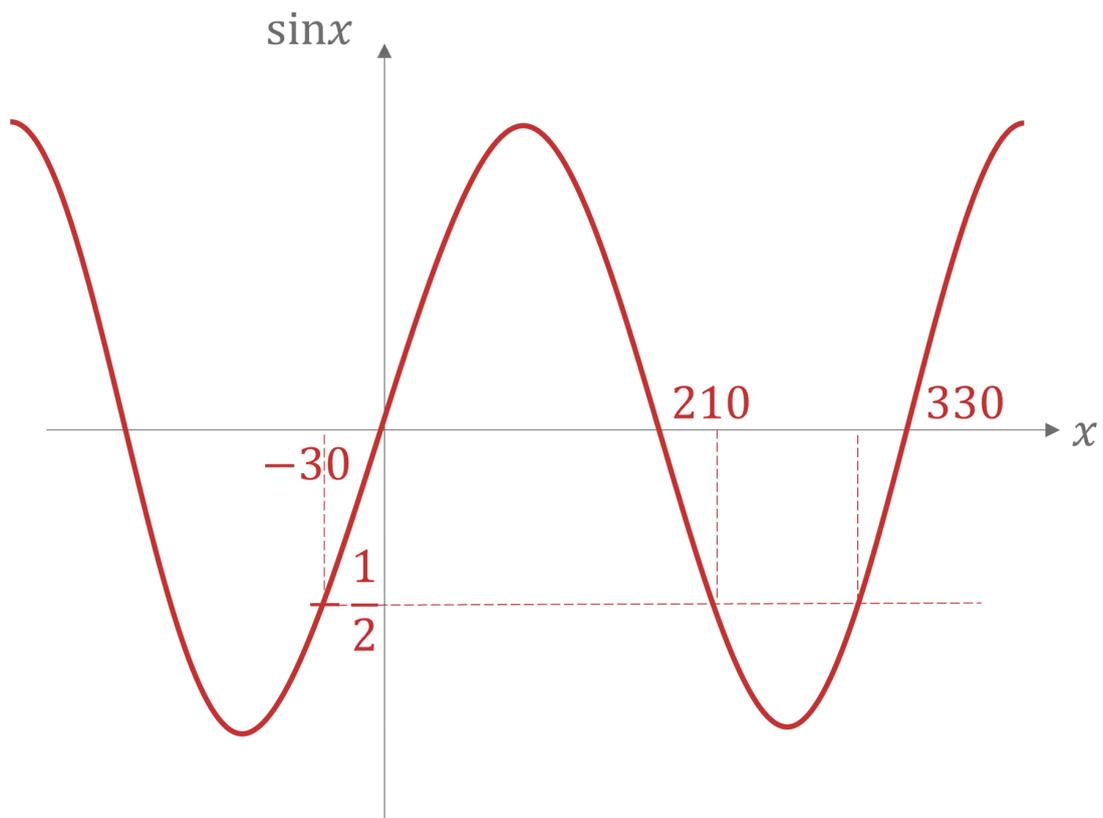$$(2\sin x + 1)(4\sin x - 3) = 0$$

[1]

Solve the first bracket equal to 0

$$2\sin x + 1 = 0$$

$$\sin x = -\frac{1}{2}$$

$$x = \sin^{-1}\left(-\frac{1}{2}\right) = -30°$$

Use a sketch of the sin graph to find all the angles in the required range
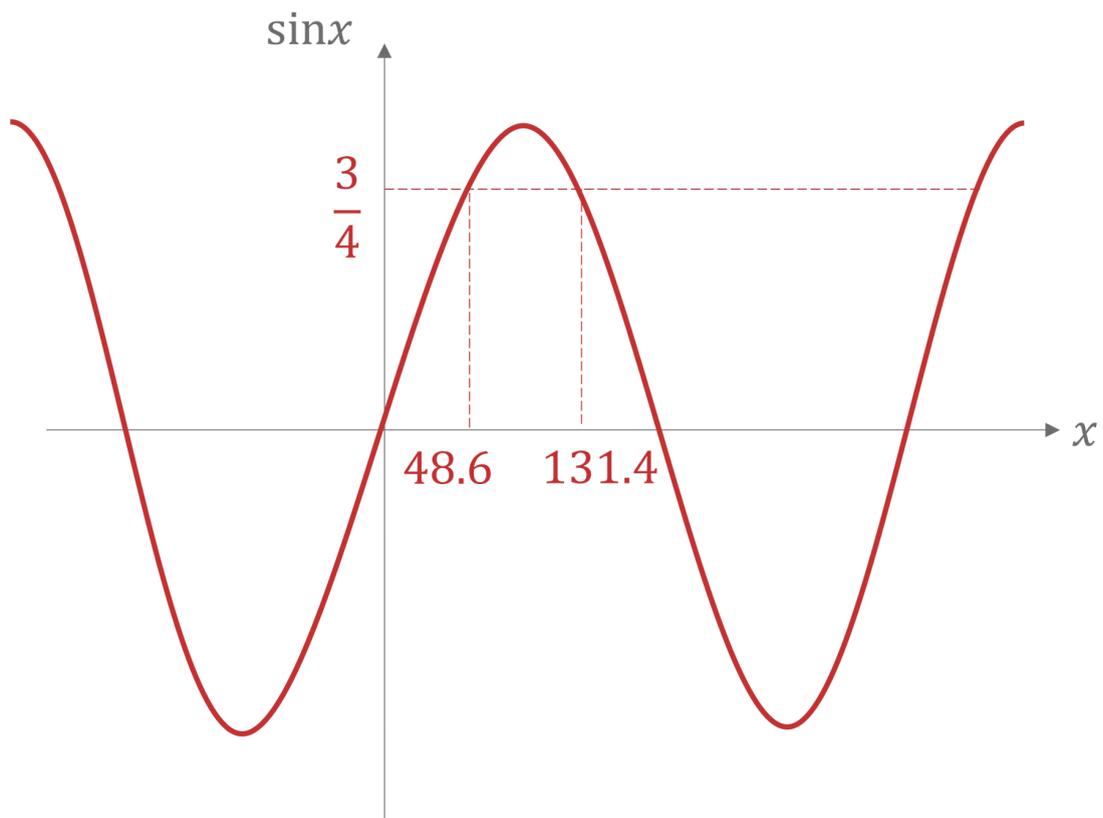

$x = 210°, \ 330°$ [1]

Solve the second bracket equal to 0

$$4\sin x - 3 = 0$$

$$\sin x = \frac{3}{4}$$

$$x = \sin^{-1}\left(\frac{3}{4}\right) = 48.590...°$$

Use a sketch of the sin graph to find all the angles in the required range

$$x = 48.6°, \ 131.4° \ \text{[1]}$$

(6 marks)

9 (a) Throughout this question, $A = \dfrac{\tan x}{1 + \sec x}$ and $B = \dfrac{\tan x}{1 - \sec x}$.

Show that $AB = k$ where k is a constant to be found.

Answer

Multiply A and B together

$$AB = \left(\frac{\tan x}{1 + \sec x} \right) \left(\frac{\tan x}{1 - \sec x} \right)$$

Expand out the brackets in the numerator and the denominator (using the difference between two squares)

$$AB = \frac{\tan^2 x}{1 - \sec^2 x}$$

Method 1

Use the identity $1 + \tan^2 x = \sec^2 x$ in the denominator and simplify

$$AB = \frac{\tan^2 x}{1 - (1 + \tan^2 x)}$$

$$= \frac{\tan^2 x}{1 - 1 - \tan^2 x}$$

$$= \frac{\tan^2 x}{-\tan^2 x}$$

$$= -1$$

$$AB = -1$$

[M1 A1]

Mark Scheme and Guidance

M1: For attempting to use $1 + \tan^2 x = \sec^2 x$ to simplify the algebraic fraction.

A1: For the correct final answer of -1.

Method 2

Use the identity $\tan x = \dfrac{\sin x}{\cos x}$ and that $\sec x = \dfrac{1}{\cos x}$, then multiply top and bottom by $\cos^2 x$

$$AB = \frac{\left(\dfrac{\sin^2 x}{\cos^2 x}\right)}{1 - \dfrac{1}{\cos^2 x}}$$

$$= \frac{\left(\dfrac{\sin^2 x}{\cos^2 x}\right) \times \cos^2 x}{\left(1 - \dfrac{1}{\cos^2 x}\right) \times \cos^2 x}$$

$$= \frac{\sin^2 x}{\cos^2 x - 1}$$

Then use the identity $\cos^2 x = 1 - \sin^2 x$ in the denominator

$$AB = \frac{\sin^2 x}{(1 - \sin^2 x) - 1}$$

$$= \frac{\sin^2 x}{-\sin^2 x}$$

$$= -1$$

$AB = -1$

[M1 A1]

Mark Scheme and Guidance

M1: For attempting to use $\sin^2 x + \cos^2 x = 1$ to simplify the algebraic fraction.

A1: For the correct final answer of -1.

(2 marks)

(b) Show that $A + B = -2\cot x$.

Answer

Method 1

Add A and B together

$$A + B = \frac{\tan x}{1 + \sec x} + \frac{\tan x}{1 - \sec x}$$

Add the algebraic fractions on the right-hand side (by putting them over the lowest common denominator)

$$A + B = \frac{\tan x(1 - \sec x)}{(1 + \sec x)(1 - \sec x)} + \frac{\tan x(1 + \sec x)}{(1 + \sec x)(1 - \sec x)}$$

$$= \frac{\tan x(1 - \sec x) + \tan x(1 + \sec x)}{(1 + \sec x)(1 - \sec x)}$$

[M1 A1]

Expand out the brackets in the numerator and the denominator (using the difference between two squares)

$$A + B = \frac{\tan x - \tan x\sec x + \tan x + \tan x\sec x}{1 - \sec^2 x}$$

$$= \frac{2\tan x}{1 - \sec^2 x}$$

[M1]

Use the identity $1 + \tan^2 x = \sec^2 x$ in the denominator and simplify

$$A + B = \frac{2\tan x}{1 - (1 + \tan^2 x)}$$

$$= \frac{2\tan x}{1 - 1 - \tan^2 x}$$

$$= \frac{2\tan x}{-\tan^2 x}$$

$$= -\frac{2}{\tan x}$$

$$= -2\cot x$$

$$A + B = -2\cot x$$

[A1]

Mark Scheme and Guidance

The last mark comes from showing how the identity leads to the answer given in the question (no marks are awarded for just writing down the "show that" answer).

Method 2

Use the identity $\tan x = \dfrac{\sin x}{\cos x}$ and that $\sec x = \dfrac{1}{\cos x}$

$$A + B = \dfrac{\dfrac{\sin x}{\cos x}}{1 + \dfrac{1}{\cos x}} + \dfrac{\dfrac{\sin x}{\cos x}}{1 - \dfrac{1}{\cos x}}$$

Then multiply top and bottom of each fraction by $\cos x$

$$A + B = \dfrac{\sin x}{\cos x + 1} + \dfrac{\sin x}{\cos x - 1}$$

Add the algebraic fractions on the right-hand side (by putting them over the lowest common denominator)

$$A + B = \dfrac{\sin x(\cos x - 1)}{(\cos x + 1)(\cos x - 1)} + \dfrac{\sin x(\cos x + 1)}{(\cos x + 1)(\cos x - 1)}$$

$$= \dfrac{\sin x(\cos x - 1) + \sin x(\cos x + 1)}{(\cos x + 1)(\cos x - 1)}$$

[M1 A1]

Expand out the brackets in the numerator and the denominator (using the difference between two squares)

$$A + B = \dfrac{\sin x\cos x - \sin x + \sin x\cos x + \sin x}{\cos^2 x - 1}$$

$$= \dfrac{2\sin x\cos x}{\cos^2 x - 1}$$

Then use the identity $\cos^2 x = 1 - \sin^2 x$ in the denominator

$$A + B = \frac{2\sin x \cos x}{(1 - \sin^2 x) - 1}$$

$$= \frac{2\sin x \cos x}{-\sin^2 x}$$

$$= \frac{2\cos x}{\sin x}$$

Now use the identity that $\cot x = \dfrac{\cos x}{\sin x}$

$$A + B = -2\cot x$$

Mark Scheme and Guidance

The last mark comes from showing how the identity leads to the answer given in the question (no marks are awarded for just writing down the "show that" answer).

(4 marks)

(c) Use your answers to part (a) and part (b) to simplify the expression $(A + 1)(B + 1)$.

Hence, or otherwise, solve

$$\left(\frac{\tan(x - 60)}{1 + \sec(x - 60)} + 1\right)\left(\frac{\tan(x - 60)}{1 - \sec(x - 60)} + 1\right) = 2\sqrt{3}$$

for $-180° < x < 180°$.

Answer

Start by simplifying $(A + 1)(B + 1)$, e.g. expand the brackets

$$(A + 1)(B + 1) = AB + A + B + 1$$

Substitute in $AB = -1$ from part (a) and $A + B = -2\cot x$ from part (b) and simplify

$$(A+1)(B+1) = -1 + (-2\cot x) + 1$$
$$= -2\cot x$$

[M1 A1]

This gives the following identity

$$\left(\frac{\tan x}{1 + \sec x} + 1\right)\left(\frac{\tan x}{1 - \sec x} + 1\right) = -2\cot x$$

Change the arguments in the identity from x to $(x - 60)$ (this does not affect the identity)

$$\left(\frac{\tan(x-60)}{1 + \sec(x-60)} + 1\right)\left(\frac{\tan(x-60)}{1 - \sec(x-60)} + 1\right) = -2\cot(x-60)$$

Use this new identity to simplify the left-hand side of the equation given in the question

$$-2\cot(x-60) = 2\sqrt{3}$$

Solve this equation by first making $\tan(x - 60)$ the subject

$$-\frac{2}{\tan(x-60)} = 2\sqrt{3}$$
$$-\frac{1}{\sqrt{3}} = \tan(x-60)$$

[A1]

Note that $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ and that $\tan(-30) = -\dfrac{\sqrt{3}}{3}$ (an exact trig value)

$$x - 60 = -30, \; -30 \pm 180, \; -30 \pm 2 \times 180, \; ...$$
$$x - 60 = -30, \; -210, \; 150, \; ...$$
$$x = -30 + 60, \; -210 + 60, \; 150 + 60, \; ...$$
$$x = 30, \; -150, \; 210, \; ...$$

Find the solutions in the range $-180° < x < 180°$

$x = 30°$ **and** $x = -150°$