IGCSE · Cambridge (CIE) · Further Maths

1 hour   12 questions

Exam Questions

Vectors in Two Dimensions

Introduction to Vectors / Vector Addition / Problem Solving using Vectors / Compose & Resolve Velocities

Medium (2 questions)	/14
Hard (5 questions)	/35
Very Hard (5 questions)	/38
Total Marks	**/87**

Medium Questions

1 (a) Find the unit vector in the direction of $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

Answer

To find a unit vector, divide the vector by its magnitude.
Find the magnitude of the vector.

$$\text{Magnitude} = \sqrt{5^2 + (-12)^2}$$
$$\text{Magnitude} = 13$$

Divide the vector by 13.

$$\frac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ -\frac{12}{13} \end{pmatrix}$$ [1]

(1 mark)

(b) Given that $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + k\begin{pmatrix} -2 \\ 3 \end{pmatrix} = r\begin{pmatrix} -10 \\ 5 \end{pmatrix}$, find the value of each of the constants k and r.

Answer

Multiply k and r by each component of the corresponding vectors.

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -2k \\ 3k \end{pmatrix} = \begin{pmatrix} -10r \\ 5r \end{pmatrix}$$

Add together the vectors.

$$\begin{pmatrix} 4-2k \\ 1+3k \end{pmatrix} = \begin{pmatrix} -10r \\ 5r \end{pmatrix}$$

Equate the x components.

$$4-2k=-10r$$

Equate the y components.

$$1+3k=5r$$

[1]

Solve the equations simultaneously.

$$\begin{array}{rr} & -4+2k=10r \\ - & 2+6k=10r \\ \hline & -6-4k=0 \\ & -4k=6 \end{array}$$

[1]

$$k=-\frac{3}{2}$$

Substitute $k=-\frac{3}{2}$ into either of the equations to find r.

$$1+3\left(-\frac{3}{2}\right)=5r$$

$$1-\frac{9}{2}=5r$$

$k=-\frac{3}{2}$, $r=-\frac{7}{10}$ [1]

(3 marks)

(c) Relative to an origin O, the points A, B and C have position vectors $\mathbf{p}$, $3\mathbf{q}-\mathbf{p}$ and $9\mathbf{q}-5\mathbf{p}$ respectively.

(i) Find $\overrightarrow{AB}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

[1]

(ii) Find $\overrightarrow{AC}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

[1]

(iii) Explain why A, B and C all lie in a straight line.

[1]

(iv) Find the ratio $AB : BC$.

[1]

Answer

(i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

$$\overrightarrow{AB} = (3\mathbf{q} - \mathbf{p}) - \mathbf{p}$$
$$\overrightarrow{AB} = 3\mathbf{q} - 2\mathbf{p}$$

$3\mathbf{q} - 2\mathbf{p}$ [1]

(ii) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$.

$$\overrightarrow{AC} = (9\mathbf{q} - 5\mathbf{p}) - \mathbf{p}$$
$$\overrightarrow{AC} = 9\mathbf{q} - 6\mathbf{p}$$

$9\mathbf{q} - 6\mathbf{p}$ [1]

(iii) Both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ share a common point, *A*, and $\overrightarrow{AC} = 3(\overrightarrow{AB})$.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ share a common point - point *A* - and since $\overrightarrow{AC} = 3(\overrightarrow{AB})$, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel [1]

(iv) Find $\overrightarrow{BC}$.

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OB}$.

$$\overrightarrow{BC} = (9\mathbf{q} - 5\mathbf{p}) - (3\mathbf{q} - \mathbf{p})$$
$$\overrightarrow{BC} = 6\mathbf{q} - 4\mathbf{p}$$
$$\overrightarrow{BC} = 2(3\mathbf{q} - 2\mathbf{p})$$

Therefore, $\overrightarrow{BC} = 2\left(\overrightarrow{AB}\right)$ and the ratio of $AB:BC$ is $1:2$.

$1:2$ [1]

(4 marks)

2 (a) The unit vectors $\mathbf{i}$ and $\mathbf{j}$ represent due east and due north respectively.

Person A starts at a position of $(-4\mathbf{i}+\mathbf{j})$ metres and walks with a constant velocity of $(3\mathbf{i}-\mathbf{j})$ metres per second.

Find the position vector of person A after 5 seconds.

Answer

For every one second, person A moves $(3\mathbf{i}-\mathbf{j})$ metres

Add five lots of $(3\mathbf{i}-\mathbf{j})$ on to the original position $(-4\mathbf{i}+\mathbf{j})$

$$-4\mathbf{i}+\mathbf{j}+5(3\mathbf{i}-\mathbf{j})$$

Expand and collect each component

$$-4\mathbf{i}+\mathbf{j}+15\mathbf{i}-5\mathbf{j}$$

$(11\mathbf{i}-4\mathbf{j})$ **metres**

[B1]

(1 mark)

(b) Person B walks with a constant velocity of $(2\mathbf{i}+2\sqrt{3}\mathbf{j})$ metres per second.

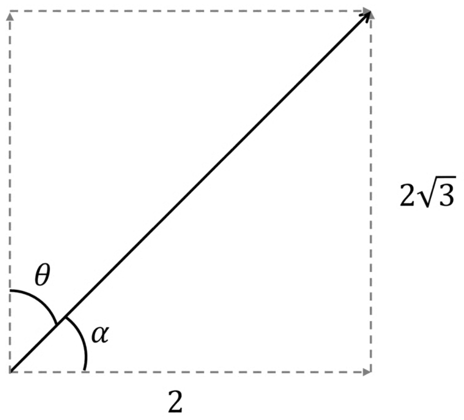
Find

(i) the speed of person B,

(ii) the bearing of the direction in which person B is walking.

Answer

It helps to sketch the velocity vector and the bearing (measured clockwise from due north)



(i)

The speed is the magnitude of the vector (the hypotenuse of the triangle) so use Pythagoras' theorem

$$\sqrt{2^2+\left(2\sqrt{3}\right)^2}$$

[M1]

Simplify

$$\sqrt{4+4\times 3}=\sqrt{16}$$

4 metres per second

[A1]

(ii)

One way to find the bearing is to first find angle α (using trigonometry)

$$\tan\alpha=\frac{2\sqrt{3}}{2}=\sqrt{3}$$

[M1]

Solving this using exact trig values

$$\alpha = 60°$$

[A1]

Hence find θ using $\theta = 90° - \alpha$

$$\theta = 30°$$

Write your final answer as a bearing (three digits needed)

$030°$

[A1]

Mark Scheme and Guidance

M1: For a correct trigonometric statement in sin, cos or tan.

A1: For finding either α or θ correctly using exact trig values.

A1: For presenting the correct angle as a 3-digit bearing.

(5 marks)

Hard Questions

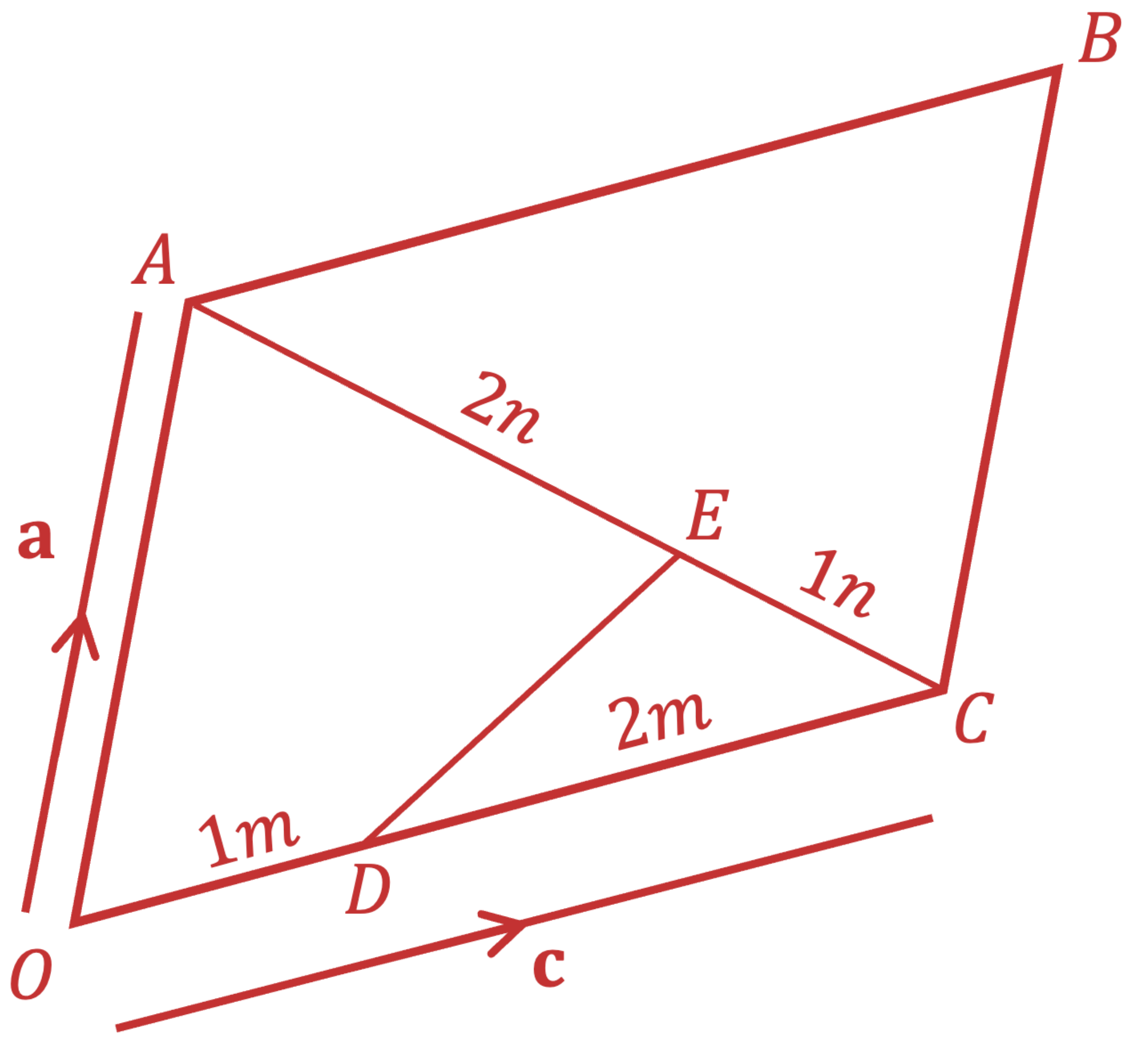
1 The parallelogram $OABC$ is such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The point D lies on OC such that $OD : DC = 1 : 2$. The point E lies on AC such that $AE : EC = 2 : 1$.

Show that $\overrightarrow{OB} = k\overrightarrow{DE}$, where k is an integer to be found.

Answer

Start by sketching a diagram with the information given. D is $\frac{1}{3}$ of the way from O

to C and E is $\frac{2}{3}$ of the way from A to C.



Write $\overrightarrow{OB}$ in terms of position vectors **a** and **c**.

$$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$$

[1]

Write a path from D to E.

$$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE}$$

Now write $\overrightarrow{DC}$ and $\overrightarrow{CE}$ in terms of position vectors **a** and **c**. Start with $\overrightarrow{DC}$.

$$\overrightarrow{DC} = \frac{2}{3}\mathbf{c}$$

[1]

Now look at $\overrightarrow{CE}$.

$$\overrightarrow{CE} = \frac{1}{3}\overrightarrow{CA}$$

Writing $\overrightarrow{CA}$ in terms of position vectors $\mathbf{a}$ and $\mathbf{c}$.

$$\overrightarrow{CA} = -\mathbf{c} + \mathbf{a}$$

Therefore,

$$\overrightarrow{CE} = \frac{1}{3}\overrightarrow{CA} = \frac{1}{3}(-\mathbf{c} + \mathbf{a})$$

[1]

Now write the path from *D* to *E* using the position vector expressions found above.

$$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE}$$
$$= \frac{2}{3}\mathbf{c} + \frac{1}{3}(-\mathbf{c} + \mathbf{a})$$

Simplify.

$$= \frac{2}{3}\mathbf{c} - \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{a}$$
$$= \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{a}$$

[1]

$$= \frac{1}{3}(\mathbf{c} + \mathbf{a})$$

Now we know that $\overrightarrow{OB} = \mathbf{c} + \mathbf{a}$ and $\overrightarrow{DE} = \frac{1}{3}(\mathbf{c} + \mathbf{a})$.

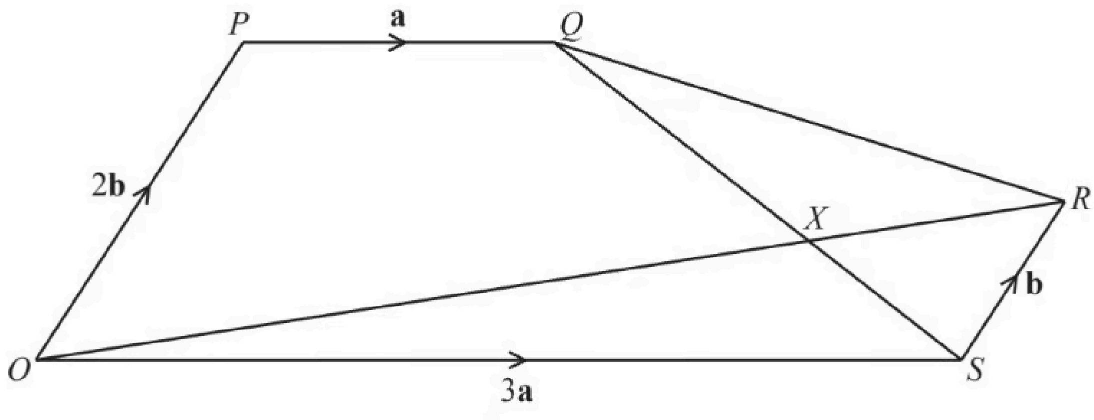
$$\overrightarrow{OB} = 3\overrightarrow{DE}$$

$k = 3$ [1]

(5 marks)

2 (a) In the diagram $\overrightarrow{OP} = 2\mathbf{b}$, $\overrightarrow{OS} = 3\mathbf{a}$, $\overrightarrow{SR} = \mathbf{b}$ and $\overrightarrow{PQ} = \mathbf{a}$. The lines OR and QS intersect at X.

Find $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



Answer

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$
$$\overrightarrow{OQ} = 2\mathbf{b} + \mathbf{a}$$

$\overrightarrow{OQ} = 2\mathbf{b} + \mathbf{a}$ [1]

(1 mark)

(b) Find $\overrightarrow{QS}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Answer

$$\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PO} + \overrightarrow{OS}$$
$$\overrightarrow{QS} = -\mathbf{a} - 2\mathbf{b} + 3\mathbf{a}$$
$$\overrightarrow{QS} = 2\mathbf{a} - 2\mathbf{b}$$

$\overrightarrow{QS} = 2\mathbf{a} - 2\mathbf{b}$ [1]

(1 mark)

(c) Given that $\overrightarrow{QX} = \mu\overrightarrow{QS}$, find $\overrightarrow{OX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and μ.

Answer

$$\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QX}$$

The question tells us that $\overrightarrow{QX} = \mu\overrightarrow{QS}$.

$$\overrightarrow{OX} = 2\mathbf{b} + \mathbf{a} + \mu\overrightarrow{QS}$$

From part (b), $\overrightarrow{QS} = 2\mathbf{a} - 2\mathbf{b}$.

$$\overrightarrow{OX} = 2\mathbf{b} + \mathbf{a} + \mu(2\mathbf{a} - 2\mathbf{b})$$

$\overrightarrow{OX} = (1 + 2\mu)\mathbf{a} + (2 - 2\mu)\mathbf{b}$ [1]

(1 mark)

(d) Given that $\overrightarrow{OX} = \lambda\overrightarrow{OR}$, find $\overrightarrow{OX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.

Answer

$$\overrightarrow{OX} = \lambda\overrightarrow{OR}$$

Find $\overrightarrow{OR}$.

$$\overrightarrow{OR} = \overrightarrow{OS} + \overrightarrow{SR}$$
$$\overrightarrow{OR} = 3\mathbf{a} + \mathbf{b}$$

Use this to find $\overrightarrow{OX}$.

$$\overrightarrow{OX} = \lambda(3\mathbf{a} + \mathbf{b})$$

$\overrightarrow{OX} = 3\lambda\mathbf{a} + \lambda\mathbf{b}$ [1]

(1 mark)

(e) Find the value of λ and of μ.

Answer

From part (c), $\overrightarrow{OX} = (1 + 2\mu)\mathbf{a} + (2 - 2\mu)\mathbf{b}$.

From part (d), $\overrightarrow{OX} = 3\lambda\mathbf{a} + \lambda\mathbf{b}$.

In the above equations, $\overrightarrow{OX}$ must be equivalent so equate both.

Compare coefficients of $\mathbf{b}$.

$$(1) \qquad 2 - 2\mu = \lambda$$

Compare coefficients of $\mathbf{a}$.

$$(2) \qquad 1 + 2\mu = 3\lambda$$

[1]

Solve equations (1) and (2) simultaneously by adding them.

$$3 = 4\lambda$$

$$\lambda = \frac{3}{4}$$

for attempting to solve [1]

Substitute $\lambda = \frac{3}{4}$ into equation (1) to find μ.

$$2 - 2\mu = \frac{3}{4}$$

$$-2\mu = -\frac{5}{4}$$

$$\mu = \frac{5}{8}$$

$\lambda = \frac{3}{4}$, $\mu = \frac{5}{8}$ [1]

(3 marks)

(f) Find the value of $\frac{QX}{XS}$.

Answer

$$\overrightarrow{QX} = \mu\overrightarrow{QS}$$

$\mu = \frac{5}{8}$ from part (e).

$$\overrightarrow{QX} = \frac{5}{8}\overrightarrow{QS}$$

Find the ratio of $QX{:}XS$.

$$QX{:}XS = 5{:}3$$

$$\frac{QX}{XS} = \frac{5}{3}$$

$\frac{QX}{XS} = \frac{5}{3}$ [1]

(1 mark)

(g) Find the value of $\frac{OR}{OX}$.

Answer

$$\overrightarrow{OR} = 3\mathbf{a} + \mathbf{b}$$

From part (d) we know that $\overrightarrow{OX} = 3\lambda\mathbf{a} + \lambda\mathbf{b}$.

From part (e) we know that $\lambda = \frac{3}{4}$.

$$\overrightarrow{OX} = \frac{9}{4}\mathbf{a} + \frac{3}{4}\mathbf{b} = \frac{3}{4}(3\mathbf{a} + \mathbf{b}) = \frac{3}{4}\overrightarrow{OR}$$

Find the ratio $OR : OX$.

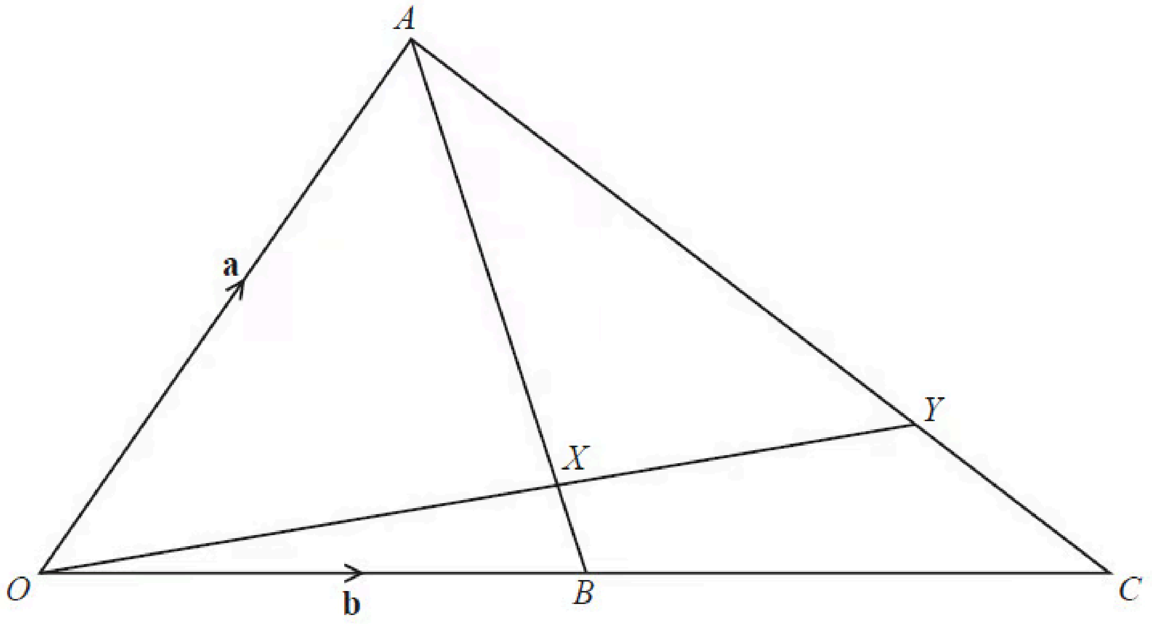
$$OR : OX = 4:3$$

$$\frac{OR}{OX} = \frac{4}{3}$$

$\frac{OR}{OX} = \frac{4}{3}$ [1]

(1 mark)

3 (a)



The diagram shows the triangle OAC. The point B is the midpoint of OC. The point Y lies on AC such that OY intersects AB at the point X where $AX : XB = 3:1$. It is given that $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.

Find $\overrightarrow{OX}$ in terms of a and b, giving your answer in its simplest form.

Answer

Find $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

[1]

We are told that $AX:XB = 3:1$, therefore $\overrightarrow{AX} = \frac{3}{4}(\overrightarrow{AB})$.

$$\overrightarrow{AX} = \frac{3}{4}(\mathbf{b} - \mathbf{a})$$

Now we can find $\overrightarrow{OX}$.

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$

$$\overrightarrow{OX} = \mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a})$$

[1]

Expand and simplify.

$\overrightarrow{OX} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$ [1]

(3 marks)

(b) Find *AC* in terms of **a** and **b**.

Answer

B is the midpoint of *OC*. Therefore, $\overrightarrow{OC} = 2\boldsymbol{b}$.

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$\overrightarrow{AC} = 2\mathbf{b} - \mathbf{a}$$

$\overrightarrow{AC} = 2\mathbf{b} - \mathbf{a}$ [1]

(1 mark)

(c) Given that $\overrightarrow{OY} = h\overrightarrow{OX}$, find $\overrightarrow{AY}$ in terms of a, b and h.

Answer

$\overrightarrow{AY} = \overrightarrow{AO} + \overrightarrow{OY}$. Since we are told that $\overrightarrow{OY} = h\overrightarrow{OX}$,

$$\overrightarrow{AY} = \overrightarrow{AO} + h\overrightarrow{OX} = h\overrightarrow{OX} - \overrightarrow{OA}$$

$$\overrightarrow{AY} = h\left(\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}\right) - \mathbf{a}$$

$\overrightarrow{AY} = \left(\frac{h-4}{4}\right)\mathbf{a} + \frac{3h}{4}\mathbf{b}$ [1]

(1 mark)

(d) Given that $\overrightarrow{AY} = m\overrightarrow{AC}$, find the value of h and of m.

Answer

Substitute $\overrightarrow{AY} = \left(\frac{h-4}{4}\right)\mathbf{a} + \frac{3h}{4}\mathbf{b}$ and $\overrightarrow{AC} = 2\boldsymbol{b} - \boldsymbol{a}$ into $\overrightarrow{AY} = m\overrightarrow{AC}$

$$\left(\frac{h-4}{4}\right)\mathbf{a} + \frac{3h}{4}\mathbf{b} = m(2\mathbf{b} - \mathbf{a})$$

Expand the brackets on the right-hand side.

$$\left(\frac{h-4}{4}\right)\mathbf{a} + \frac{3h}{4}\mathbf{b} = -m\mathbf{a} + 2m\mathbf{b}$$

[1]

Equate $\mathbf{a}$ and $\mathbf{b}$ vectors to form two equations in h and m.

$$\frac{h-4}{4} = -m$$

[1]

$$\frac{3h}{4} = 2m$$

[1]

Solve simultaneously.

$$m = \frac{3h}{8}$$

$$\therefore \frac{h-4}{4} = -\frac{3h}{8}$$

$$8h - 32 = -12h$$

$$20h = 32$$

$$h = \frac{32}{20} = \frac{8}{5}$$

Substitute into either of the original equations and solve for m.

$$\frac{3}{4}\left(\frac{8}{5}\right)=2m$$

$$m=\frac{24}{40}=\frac{3}{5}$$

$h=\frac{8}{5}$ and $m=\frac{3}{5}$ [1]

(4 marks)

4 (a) The vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a} = \alpha\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 12\mathbf{i} + \beta\mathbf{j}$.

Find the value of each of the constants α and β such that $4\mathbf{a} - \mathbf{b} = (\alpha + 3)\mathbf{i} - 2\mathbf{j}$.

Answer

Substitute $\mathbf{a} = \alpha\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 12\mathbf{i} + \beta\mathbf{j}$ into $4\mathbf{a} - \mathbf{b} = (\alpha + 3)\mathbf{i} - 2\mathbf{j}$.

$$4(\alpha\mathbf{i} + \mathbf{j}) - (12\mathbf{i} + \beta\mathbf{j}) = (\alpha + 3)\mathbf{i} - 2\mathbf{j}$$

Expand the brackets.

$$4\alpha\mathbf{i} + 4\mathbf{j} - 12\mathbf{i} - \beta\mathbf{j} = (\alpha + 3)\mathbf{i} - 2\mathbf{j}$$

Equate the coefficients of $\mathbf{i}$.

$$4\alpha - 12 = \alpha + 3$$

$$3\alpha = 15$$

$$\alpha = 5$$

Equate the coefficients of $\mathbf{j}$.

$$4 - \beta = -2$$

method for both α and β [1]

$$6 = \beta$$

$\boldsymbol{\alpha = 5}$ [1]
$\boldsymbol{\beta = 6}$ [1]
(3 marks)

(b) Hence find the unit vector in the direction of $\mathbf{b} - 4\mathbf{a}$.

Answer

First find $\mathbf{b} - 4\mathbf{a}$.

$$\mathbf{b} - 4\mathbf{a} = (12\mathbf{i} + \beta\mathbf{j}) - 4(\alpha\mathbf{i} + \mathbf{j})$$

Use $\alpha = 5$ and $\beta = 6$ from part (a).

$$\mathbf{b} - 4\mathbf{a} = (12\mathbf{i} + 6\mathbf{j}) - 4(5\mathbf{i} + \mathbf{j})$$
$$\mathbf{b} - 4\mathbf{a} = -8\mathbf{i} + 2\mathbf{j}$$

To find the unit vector, divide by the magnitude.

$$\sqrt{(-8)^2 + (2)^2} = \sqrt{68} = 2\sqrt{17}$$

[1]

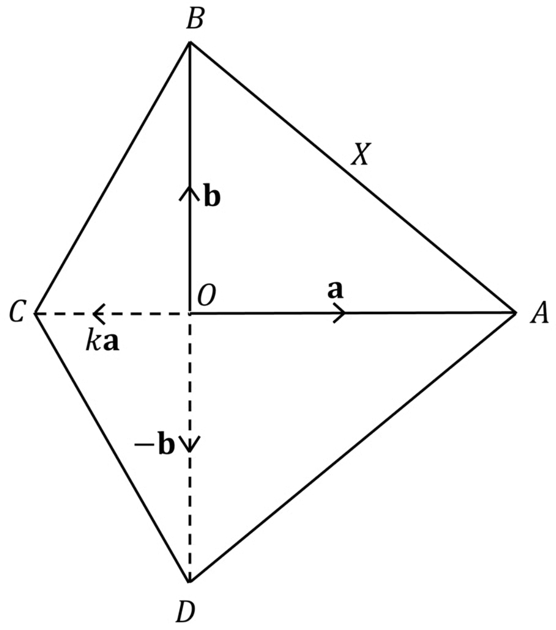
$$\text{Unit vector} = \frac{1}{2\sqrt{17}}(-8\mathbf{i} + 2\mathbf{j})$$

Unit vector in the direction of $\mathbf{b} - 4\mathbf{a}$ is $\frac{\sqrt{17}}{17}(-4\mathbf{i} + 2\mathbf{j})$ [1]

(2 marks)

5 (a) The kite $ABCD$ has diagonals CA and BD which intersect at O, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = k\mathbf{a}$ and $\overrightarrow{OD} = -\mathbf{b}$.

The point X is the midpoint of AB, as shown.



Find

(i) $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

(ii) $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

(iii) $\overrightarrow{CD}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$

Answer

(i)

E.g. travel from *A* to *O* to *B*

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= \left(-\overrightarrow{OA}\right) + \overrightarrow{OB} \\ &= -\mathbf{a} + \mathbf{b}\end{aligned}$$

$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$

[B1]

(ii)

Use your answer from part (i) to help

E.g. travel from *O* to *A* then *A* to *X* (which uses half of *A* to *B*)

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX}\\ &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}\\ &= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})\end{aligned}$$

$\overrightarrow{OX} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

[B1]

(iii)

E.g. travel from *C* to *O* then *O* to *D*

$$\begin{aligned}\overrightarrow{CD} &= \overrightarrow{CO} + \overrightarrow{OD}\\ &= \left(-\overrightarrow{OC}\right) + \overrightarrow{OD}\\ &= -k\mathbf{a} + (-\mathbf{b})\\ &= -k\mathbf{a} - \mathbf{b}\end{aligned}$$

$\overrightarrow{CD} = -k\mathbf{a} - \mathbf{b}$

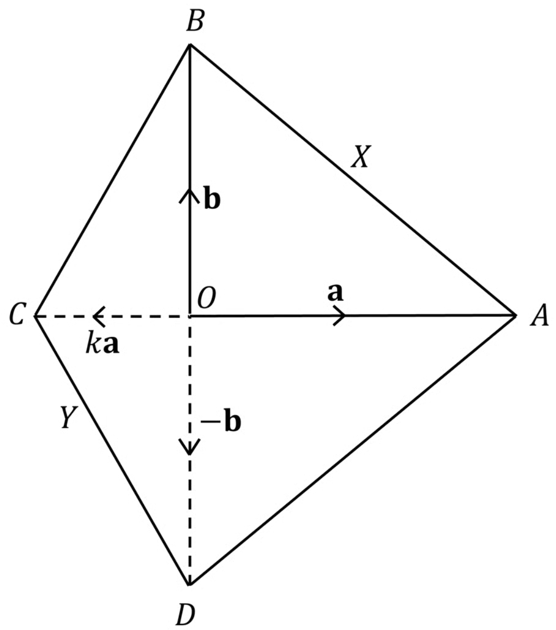
[B1]
(3 marks)

(b) The point Y lies on CD such that $CY : YD = 1:2$.

If $\overrightarrow{YX}$ is parallel to $\mathbf{a} + \mathbf{b}$, find k.

Answer

Add Y to the diagram



You found vector $\overrightarrow{CD} = -k\mathbf{a} - \mathbf{b}$ in part (a)(iii)

If Y splits the line CD into the ratio $1:2$ (1 + 2 = 3 parts to the ratio) then $\overrightarrow{YD}$ is $\frac{2}{3}$ of $\overrightarrow{CD}$

$$\overrightarrow{YD} = \frac{2}{3}\overrightarrow{CD}$$

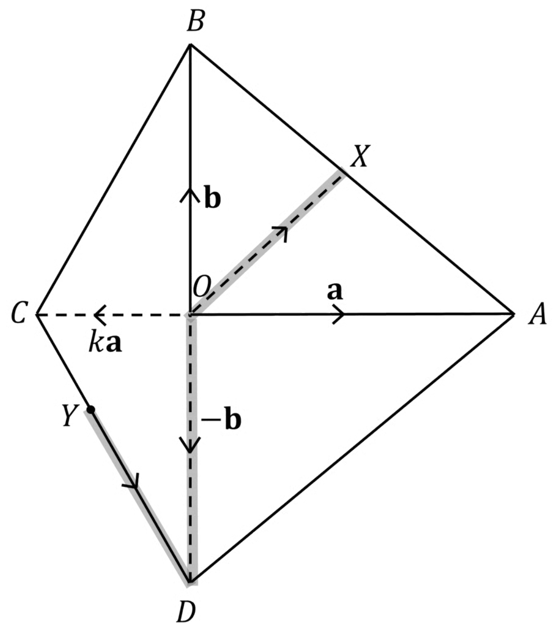
$$= \frac{2}{3}(-k\mathbf{a} - \mathbf{b})$$

[B1]

Mark Scheme and Guidance

This mark is for either $\overrightarrow{YD} = \frac{2}{3}(-k\mathbf{a} - \mathbf{b})$ or $\overrightarrow{YC} = -\frac{1}{3}(-k\mathbf{a} - \mathbf{b})$.

Now find $\overrightarrow{YX}$ e.g. by going from *Y* to *D* then *D* to *O* then *O* to *X* (using part (a)(ii))



$$\overrightarrow{YX} = \overrightarrow{YD} + \overrightarrow{DO} + \overrightarrow{OX}$$
$$= \overrightarrow{YD} + \left(-\overrightarrow{OD}\right) + \overrightarrow{OX}$$
$$= \frac{2}{3}(-k\mathbf{a} - \mathbf{b}) + (- - \mathbf{b}) + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

[M1]

Expand and simplify, collecting into $\mathbf{a}$ and $\mathbf{b}$ components

$$\overrightarrow{YX} = \left(-\frac{2}{3}k + \frac{1}{2}\right)\mathbf{a} + \left(-\frac{2}{3} + 1 + \frac{1}{2}\right)\mathbf{b}$$
$$= \left(-\frac{2}{3}k + \frac{1}{2}\right)\mathbf{a} + \frac{5}{6}\mathbf{b}$$

[A1]

If $\overrightarrow{YX}$ is meant to be parallel to $\mathbf{a} + \mathbf{b}$ then the $\mathbf{a}$ and $\mathbf{b}$ components in the expression above must be equal

$$-\frac{2}{3}k + \frac{1}{2} = \frac{5}{6}$$

Solve for k

$$-\frac{2}{3}k = \frac{1}{3}$$

$k = -\frac{1}{2}$

[A1]

Examiner Tips and Tricks

A negative k is expected as $\overrightarrow{OC} = k\mathbf{a}$ is the opposite direction to $\overrightarrow{OA} = \mathbf{a}$.

(4 marks)

Very Hard Questions

1 (a) A particle P is initially at the point with position vector $\begin{pmatrix} 30 \\ 10 \end{pmatrix}$ and moves with a constant speed of $10\ \text{ms}^{-1}$ in the same direction as $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Find the position vector of P after t s.

Answer

Speed is given by the magnitude of the velocity so the first step is to work out the magnitude of the direction vector. The magnitude of vector $|\boldsymbol{a}| = \left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$ is given by $\sqrt{x^2 + y^2}$.

$$\sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

[1]

We are told that the speed is $10\ \text{m s}^{-1}$ so we want a velocity vector that is a scalar multiple of $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ but with a magnitude of 10. 10 is double 5 so we need to double the vector $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$, therefore the velocity vector is,

$$2 \times \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$$

[1]

We then need to multiply the velocity by time and add to the initial position vector to find the position vector P after *t* seconds, therefore

$$\begin{pmatrix} 30 \\ 10 \end{pmatrix} + \begin{pmatrix} -8 \\ 6 \end{pmatrix}t$$

The position vector of P after t seconds is $\begin{pmatrix} 30 - 8t \\ 6t + 10 \end{pmatrix}$ m [1]

(3 marks)

(b) As P starts moving, a particle Q starts to move such that its position vector after t s is given by $\begin{pmatrix} -80 \\ 90 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

Write down the speed of Q.

Answer

The speed is given by the magnitude of the velocity. Velocity is $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$. (The vector $\begin{pmatrix} -80 \\ 90 \end{pmatrix}$ is the initial position.)

$$\sqrt{5^2 + 12^2} = \sqrt{169}$$

13 m s^{-1} [1]

(1 mark)

(c) Find the exact distance between P and Q when $t = 10$, giving your answer in its simplest surd form.

Answer

Substituting $t = 10$ into the position vectors for P and Q.

$$\overrightarrow{OP} = \begin{pmatrix} 30 \\ 10 \end{pmatrix} + \begin{pmatrix} -8 \\ 6 \end{pmatrix} \times 10$$

$$\overrightarrow{OP} = \begin{pmatrix} 30 \\ 10 \end{pmatrix} + \begin{pmatrix} -80 \\ 60 \end{pmatrix} = \begin{pmatrix} -50 \\ 70 \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} -80 \\ 90 \end{pmatrix} + \begin{pmatrix} 5 \\ 12 \end{pmatrix} \times 10$$

$$\overrightarrow{OQ} = \begin{pmatrix} -80 \\ 90 \end{pmatrix} + \begin{pmatrix} 50 \\ 120 \end{pmatrix} = \begin{pmatrix} -30 \\ 210 \end{pmatrix}$$

[1]

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP}.$

$$\begin{pmatrix} -30 \\ 210 \end{pmatrix} - \begin{pmatrix} -50 \\ 70 \end{pmatrix} = \begin{pmatrix} 20 \\ 140 \end{pmatrix}$$

The distance between two points is the magnitude of the vector between them.

$$\sqrt{20^2 + 140^2} = \sqrt{20000}$$

[1]

$100\sqrt{2}$ [1]

(3 marks)

2 (a) Relative to an origin O, the position vectors of the points A, B, C and D are

$$\overrightarrow{OA} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \overrightarrow{OD} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

Find the unit vector in the direction of $\overrightarrow{AB}$.

Answer

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

$$\overrightarrow{AB} = \begin{pmatrix} 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

[1]

Work out the magnitude of $\overrightarrow{AB}$ using Pythagoras' Theorem.

$$\sqrt{4^2 + 8^2}$$

[1]

$$= \sqrt{80} = 4\sqrt{5}$$

Divide the vector by its magnitude and simplify.

$$\frac{1}{4\sqrt{5}}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\frac{\sqrt{5}}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ [1]

(3 marks)

(b) The point A is the mid-point of BC. Find the value of x and of y.

Answer

Because point *A* is a midpoint of *BC*,

$$\overrightarrow{OA} = \frac{1}{2} \times \left(\overrightarrow{OB} + \overrightarrow{OC}\right)$$

$$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \frac{1}{2} \times \begin{pmatrix} 10 + x \\ 3 + y \end{pmatrix}$$

[1]

Multiply both sides by 2.

$$\begin{pmatrix} 12 \\ -10 \end{pmatrix} = \begin{pmatrix} 10 + x \\ 3 + y \end{pmatrix}$$

Therefore,

$$12 = 10 + x \quad -10 = 3 + y$$

$\boldsymbol{x = 2}$, $\boldsymbol{y = -13}$

both [1]

(2 marks)

(c) The point E lies on OD such that $OE : OD$ is $1 : 1 + \lambda$. Find the value of λ such that $\overrightarrow{BE}$ is parallel to the x-axis.

Answer

$OE : OD = 1 : 1 + \lambda$, therefore,

$$\frac{\overrightarrow{OE}}{\overrightarrow{OD}} = \frac{1}{1 + \lambda}$$

Rearrange.

$$\overrightarrow{OE} = \frac{1}{1 + \lambda} \overrightarrow{OD}$$

Substitute in the known values.

$$\overrightarrow{OE}=\frac{1}{1+\lambda}\begin{pmatrix}12\\7\end{pmatrix}$$

[1]

Find $\overrightarrow{BE}$ in terms of λ.

$$\overrightarrow{BE}=\overrightarrow{OE}-\overrightarrow{OB}$$

$$\overrightarrow{BE}=\frac{1}{1+\lambda}\begin{pmatrix}12\\7\end{pmatrix}-\begin{pmatrix}10\\3\end{pmatrix}$$

$$\overrightarrow{BE}=\begin{pmatrix}\frac{12}{1+\lambda}\\\frac{7}{1+\lambda}\end{pmatrix}-\begin{pmatrix}10\\3\end{pmatrix}$$

$$\overrightarrow{BE}=\begin{pmatrix}\frac{12}{1+\lambda}-10\\\frac{7}{1+\lambda}-3\end{pmatrix}$$

Since $\overrightarrow{BE}$ is parallel to the x axis, the y component of the vector will be 0.

$$\frac{7}{1+\lambda}-3=0$$

$$\frac{7}{1+\lambda}=3$$

[1]

Rearrange and solve.

$$7=3(1+\lambda)$$

$$7=3+3\lambda$$

$$4=3\lambda$$

$$\lambda = \frac{4}{3}$$ [1]

(3 marks)

3 (a) In this question all distances are in km.

A ship P sails from a point A, which has position vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, with a speed of 52 kmh^{-1} in the direction of $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$

Find the velocity vector of the ship.

Answer

Use "Velocity = Speed x Direction Unit Vector".

$$\text{Velocity} = 52 \times \frac{1}{\sqrt{(-5)^2 + 12^2}} \begin{pmatrix} -5 \\ 12 \end{pmatrix}$$

$$\text{Velocity} = \frac{52}{13} \begin{pmatrix} -5 \\ 12 \end{pmatrix}$$

$$\textbf{Velocity} = \begin{pmatrix} -20 \\ 48 \end{pmatrix} \textbf{km h}^{-1}$$ [1]

(1 mark)

(b) Write down the position vector of P at a time t hours after leaving A.

Answer

The position vector of *P* at a time *t* hours after leaving *A* will be given by velocity x time (since *A* is the origin).

Use the velocity found in part (a).

Position vector of *P* at time *t* hours after leaving *A* is $\begin{pmatrix} -20t \\ 48t \end{pmatrix}$ **km** [1]

(1 mark)

(c) At the same time that ship P sails from A, a ship Q sails from a point B, which has position vector $\begin{pmatrix} 12 \\ 8 \end{pmatrix}$, with velocity vector $\begin{pmatrix} -25 \\ 45 \end{pmatrix} \text{kmh}^{-1}$.

Write down the position vector of Q at a time t hours after leaving B.

Answer

Point B is **not** the origin so B's initial position vector will need to be added to its "velocity vector x time".

$$\begin{pmatrix} -25t \\ 45t \end{pmatrix} + \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

The position vector of Q at time t is $\begin{pmatrix} 12-25t \\ 45t+8 \end{pmatrix}$ km [1]

(1 mark)

(d) Using your answers to parts (b) and (c), find the displacement vector $\overrightarrow{PQ}$ at time t hours.

Answer

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

Use the answers from parts (b) and (c).

$$\overrightarrow{PQ} = \begin{pmatrix} 12-25t \\ 45t+8 \end{pmatrix} - \begin{pmatrix} -20t \\ 48t \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 12-5t \\ 8-3t \end{pmatrix}$$ [1]

(1 mark)

(e) Hence show that $PQ = \sqrt{34t^2 - 168t + 208}$.

Answer

Find, algebraically, the magnitude of your answer to part (d).

$$\left|\overrightarrow{PQ}\right| = \sqrt{(12-5t)^2 + (8-3t)^2}$$

[1]

$$= \sqrt{144 - 120t + 25t^2 + 64 - 48t + 9t^2}$$

$\sqrt{34t^2 - 168t + 208}$ [1]

(2 marks)

(f) Find the value of t when P and Q are first 2 km apart.

Answer

When *P* and *Q* are first 2 km apart, the magnitude of $\overrightarrow{PQ}$ will be 2.

$$\sqrt{34t^2 - 168t + 208} = 2$$

Solve, starting by squaring both sides.

$$34t^2 - 168t + 208 = 4$$

Make the quadratic equal to 0.

$$34t^2 - 168t + 204 = 0$$

[1]

Solve using the quadratic formula and/or calculator.

$$t = 2.148..., \ t = 2.792...$$

We want the *first* time that they are 2 km apart.

***P* and *Q* are first 2 km apart after 2.15 seconds (3 s.f.)** [1]

(2 marks)

4 The position vectors of three points, A, B and C, relative to an origin O, are $\begin{pmatrix} -5 \\ -7 \end{pmatrix}$, $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ respectively. Given that $\overrightarrow{AC} = 4\overrightarrow{BC}$, find the unit vector in the direction of $\overrightarrow{OC}$.

Answer

Since $\overrightarrow{AC}$ is a multiple of $\overrightarrow{BC}$, we know that A, B and C must be collinear. Work out $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 10 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$$

Use $\overrightarrow{AC} = 4\overrightarrow{BC}$ to find the ratio of *AB* to *BC*.

$$AB : BC = 3:1$$

[1]

$$\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AB} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Now find $\overrightarrow{OC}$.

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$

$$\overrightarrow{OC} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \end{pmatrix}$$

[1]

A unit vector in the direction of $\overrightarrow{OC}$ is required so we will need its magnitude.

$$\left|\overrightarrow{OC}\right| = \sqrt{15^2 + (-3)^2} = \sqrt{234} = 3\sqrt{26}$$

[1]

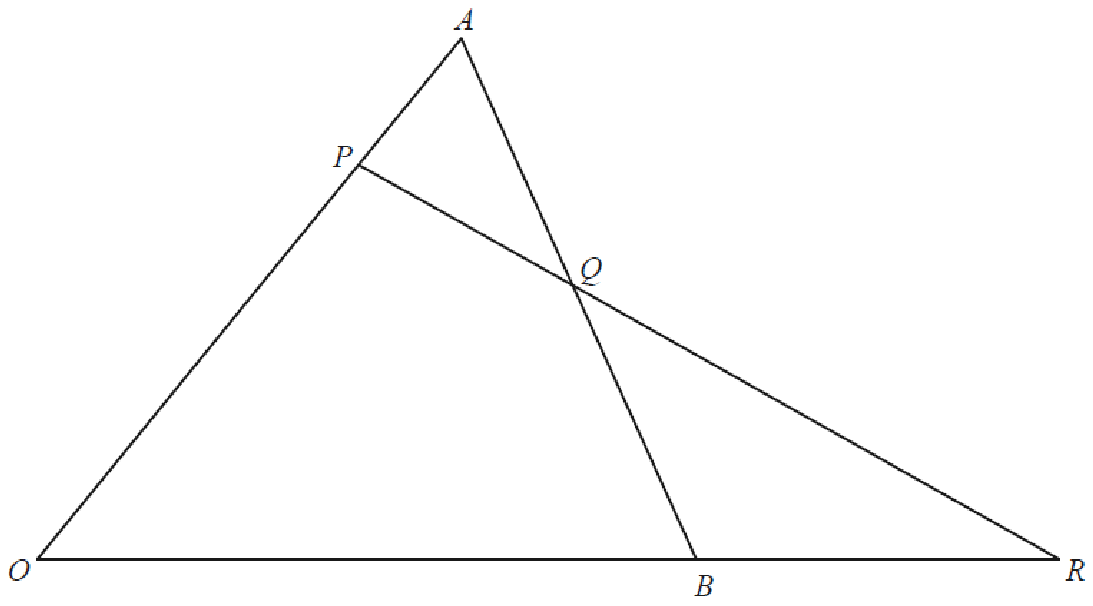
Find the unit vector by dividing $\overrightarrow{OC}$ by its magnitude.

$$\frac{1}{3\sqrt{26}}\begin{pmatrix} 15 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{26}}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$\frac{\sqrt{26}}{26}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ [1]

(5 marks)

5 (a)



The diagram shows a triangle OAB such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P lies on OA such that $OP = \frac{3}{4}OA$. The point Q is the mid-point of AB. The lines OB and PQ are extended to meet at the point R. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,

Find $\overrightarrow{AB}$

Answer

The route from *A* to *B*, in known vectors, is via *O*.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$\overrightarrow{OB} = \mathbf{b} - \mathbf{a}$ [1]

(1 mark)

(b) Find $\overrightarrow{PQ}$. Give your answer in its simplest form.

Answer

$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$$

Since $OP = \frac{3}{4}OA$, we know that $OP : PA = 3 : 1$. Find $\overrightarrow{PA}$.

$$\overrightarrow{PA} = \frac{1}{4}\overrightarrow{OA} = \frac{1}{4}\mathbf{a}$$

[1]

Point Q is the midpoint of AB. Find $\overrightarrow{AQ}$.

$$\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB}$$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ from part (a).

$$\overrightarrow{AQ} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

[1]

Add these together to find $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$$

$$\overrightarrow{PQ} = \frac{1}{4}\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

Expand and simplify.

$$\overrightarrow{PQ} = \frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$$

$\overrightarrow{PQ} = \frac{1}{2}\mathbf{b} - \frac{1}{4}\mathbf{a}$ [1]

(3 marks)

(c) It is given that $n\overrightarrow{PQ} = \overrightarrow{QR}$ and $\overrightarrow{BR} = k\mathbf{b}$, where n and k are positive constants.

Find $\overrightarrow{QR}$ in terms of n, $\mathbf{a}$ and $\mathbf{b}$.

Answer

$$\overrightarrow{QR} = n\overrightarrow{PQ}$$

From part (b) we know that $\overrightarrow{PQ} = \frac{1}{2}\mathbf{b} - \frac{1}{4}\mathbf{a}$.

$$\overrightarrow{QR} = n\left(\frac{1}{2}\mathbf{b} - \frac{1}{4}\mathbf{a}\right)$$

$\overrightarrow{QR} = n\left(\frac{1}{2}\mathbf{b} - \frac{1}{4}\mathbf{a}\right)$ [1]

(1 mark)

(d) Find $\overrightarrow{QR}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

Answer

To involve k, consider $\overrightarrow{QR}$ involving $\overrightarrow{BR}$.

$$\overrightarrow{QR} = \overrightarrow{QB} + \overrightarrow{BR}$$

Use $\overrightarrow{BR} = k\mathbf{b}$.

$$\overrightarrow{QR} = \overrightarrow{QB} + k\mathbf{b}$$

Find $\overrightarrow{QB}$.

$$\overrightarrow{QB} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

Find $\overrightarrow{QR}$.

$$\overrightarrow{QR} = \overrightarrow{QB} + k\mathbf{b}$$

[1]

$$\overrightarrow{QR} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) + k\mathbf{b}$$

$\overrightarrow{QR} = \left(k + \frac{1}{2}\right)\mathbf{b} - \frac{1}{2}\mathbf{a}$ [1]

(2 marks)

(e) Hence find the value of n and of k.

Answer

Parts (c) and (d) must be equal as both are $\overrightarrow{QR}$.

$$n\left(\frac{1}{2}\mathbf{b}-\frac{1}{4}\mathbf{a}\right)=\frac{1}{2}(\mathbf{b}-\mathbf{a})+k\mathbf{b}$$

Equate coefficients of $\mathbf{b}$.

$$\frac{1}{2}n=\frac{1}{2}+k$$

Equate coefficients of $\mathbf{a}$.

$$-\frac{1}{4}n=-\frac{1}{2}$$

[1]

Solve the second equation to find n.

$$n=2$$

Substitute $n=2$ into the first equation to find k.

$$\frac{1}{2}\times 2=\frac{1}{2}+k$$

$$k=\frac{1}{2}$$

$\mathbf{n=2}$ [1]

$\mathbf{k=\frac{1}{2}}$ [1]

(3 marks)